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THE RADIATION FIELD IN A FLUID IN MOTION

By L. H. THOMAS

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1. Introduction. The interaction between radiation and fluid in internal relative motion has been treated from various points of view and with various objects by Jeans,* Eddington,† Rosseland,‡ Vogt,§ and Milne.|| These authors have usually neglected explicitly or tacitly small terms arising from the finite velocity of light. It is not difficult to extend the treatment to include such terms exactly, and it then appears that the terms neglected were sometimes of the same order of magnitude as those retained. This arises mainly since the differences in the radiation-intensity I_ν referred to a fixed coordinate-system and referred to a coordinate-system moving with the matter, due to aberration, Doppler effect, and the transformation of the energy-density as a component of the stress-energy tensor, are all of the same order of magnitude.

In the sequel equation-systems invariant under Lorentz transformations will be obtained which take these effects exactly into account.

2. Notation. The material stress-energy tensor and equations of motion. Of three notations,

(*a*) the general relativity notation with constant fundamental tensor,

$$g_{\mu\nu} = \begin{Bmatrix} -c^2, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{Bmatrix},$$

(*b*) the usual vector notation with $(\mathbf{a}\,.\,\mathbf{b})$ for a scalar product and I for the unit dyadic,

(*c*) ordinary Cartesian notation,

that one will be used which is, at any stage, most convenient. Nota-

* *Monthly Notices*, 85 (1925), 917; 86 (1926), 328; 86 (1926), 444.

† *Monthly Notices*, 79 (1918), 13; *Internal Constitution of the Stars*, p. 90.

‡ *Astrophysical Journal*, 63, 342.

§ *Ast. Nach.* 5545 (1928), 232.

|| *Quart. J. of Math.* (Oxford) 1 (1930), 1; *Monthly Notices*, 89 (1929), 518.

tion (a) is most useful for expressing general equations of motion, but is conveniently replaced by (b) for dealing with radiation-intensity, while (c) is convenient for special cases and some proofs.

The ordinary material velocity will be denoted by

$$(u, v, w) = \mathbf{v},$$

so that

$$(V^4, V^1, V^2, V^3) = \frac{(1, u, v, w)}{\{1-(u^2+v^2+w^2)/c^2\}^{\frac{1}{2}}} = \frac{(1, \mathbf{v})}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}$$

defines the contravariant velocity four-vector V^μ. Likewise

$$\left(\frac{\partial\phi}{\partial x_4}, \frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right) = \left(\frac{\partial\phi}{\partial t}, \frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = (\dot{\phi}, \nabla\phi),$$

where ϕ is an invariant, defines the covariant gradient of ϕ, $\dfrac{\partial\phi}{\partial x_\mu}$.

For a perfect fluid, that is if we neglect self-diffusion, conduction of heat, and viscosity, the state is specified by the velocity $\mathbf{v}$, the temperature T, and the invariant particle-density ρ_0, i.e. the particle-density referred to a coordinate-system in which $\mathbf{v}$ is zero at the point considered; the pressure p and invariant energy per unit-number of particles U being functions of ρ_0 and T. The mass-density referred to such a coordinate-system is then

$$\rho_{00} = \rho_0(1+U/c^2),$$

and if

$$\rho_{000} = \rho_{00}+p/c^2$$

the particle-density and flow-vector is $\rho_0 V^\mu$ and the material stress-energy tensor is $\rho_{000} V^\mu V^\nu + pg^{\mu\nu}$.

The five hydrodynamical equations are, then,

(1) the equation of continuity or conservation of particles,

$$\frac{\partial}{\partial x_\mu}(\rho_0 V^\mu) = 0 \tag{1}$$

or, if

$$\frac{D}{Dt} = V^\mu \frac{\partial}{\partial x_\mu} = \frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right),$$

$$\frac{1}{\rho_0}\frac{D\rho_0}{Dt} = -\frac{\partial V^\mu}{\partial x_\mu} = -\frac{\partial}{\partial t}\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}} - \nabla . \frac{\mathbf{v}}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}; \tag{1.1}$$

(2) the four equations of conservation of energy and momentum,

$$\frac{\partial}{\partial x_\mu}(\rho_{000} V^\mu V^\nu + pg^{\mu\nu}) = F^\nu, \tag{2}$$

where
$$F^{\mu} = (T/c^2, \mathbf{F}) = (T/c^2, X, Y, Z),$$
and T, X, Y, Z are the rates of increase of the energy and momentum of the matter per unit-volume.

If $\rho_1 = \rho_{000}/(1-\mathbf{v}^2/c^2)$, equations (2) take the more familiar form

$$\frac{\partial}{\partial t}\left(\rho_1 - \frac{p}{c^2}\right) + \frac{\partial}{\partial x}\rho_1 u + \frac{\partial}{\partial y}\rho_1 v + \frac{\partial}{\partial z}\rho_1 w = T/c^2$$

$$\frac{\partial}{\partial t}\rho_1 u + \frac{\partial}{\partial x}(\rho_1 u^2 + p) + \frac{\partial}{\partial y}\rho_1 uv + \frac{\partial}{\partial z}\rho_1 uw = X$$

$$\frac{\partial}{\partial t}\rho_1 v + \frac{\partial}{\partial x}\rho_1 vu + \frac{\partial}{\partial y}(\rho_1 v^2 + p) + \frac{\partial}{\partial z}\rho_1 vw = Y$$

$$\frac{\partial}{\partial t}\rho_1 w + \frac{\partial}{\partial x}\rho_1 wu + \frac{\partial}{\partial y}\rho_1 wv + \frac{\partial}{\partial z}(\rho_1 w^2 + p) = Z, \tag{2.1}$$

or, more shortly,

$$\frac{\partial}{\partial t}\left(\rho_1 - \frac{p}{c^2}\right) + \nabla \cdot \rho_1 \mathbf{v} = T/c^2$$

$$\frac{\partial}{\partial t}\rho_1 \mathbf{v} + \nabla \cdot \{\rho_1 \mathbf{v}\mathbf{v} + p\mathbf{I}\} = \mathbf{F} \tag{2.2}^*$$

If equations (2) are multiplied by $-g_{\nu\sigma}V^{\sigma}$ and summed, we obtain, since $g_{\nu\sigma}V^{\nu}V^{\sigma} = -c^2$,

$$c^2 \frac{\partial}{\partial x_{\mu}} \rho_{000} V^{\mu} - V^{\mu} \frac{\partial p}{\partial x_{\mu}} = -g_{\nu\sigma} F^{\nu} V^{\sigma},$$

or, subtracting $c^2 + U + p/\rho_0$ times (1), and writing T_0 for the value of T in a coordinate-system in which $\mathbf{v}$ is zero at the point considered,

$$\rho_0 \frac{D}{Dt} U - \frac{p}{\rho_0} \frac{D\rho_0}{Dt} = T_0. \tag{3}$$

For ordinary body-forces (such as gravitation, apart from terms due to general relativity) $T_0 = 0$, and equation (3) is the one that gives most simply the change of temperature at a point,

The above has been written out to illustrate the notation; while the terms depending on c are not usually required for matter, corresponding terms for radiation may be important and it is convenient to have the material equations in a form invariant for Lorentz transformations.

* See Eddington, *Mathematical Theory of Relativity*, 116, 121, where, however, ρ_0 is used with a different meaning.

3. The effect of a Lorentz transformation on radiation intensity; Schwarzschild's equation. The intensity of radiation I_ν of frequency ν at x,y,z at time t in direction with direction-cosines

$$\mathbf{l} = (l, m, n) = (\cos\theta, \sin\theta\cos\phi, \sin\theta\sin\phi)$$

is a function of x,y,z,t,ν,θ,ϕ such that the radiant energy of frequency between ν and $\nu+d\nu$ moving in a direction within solid angle $d\omega$ about $\mathbf{l}$ through area $d\mathbf{S}$ in time dt is in the limit

$$I_\nu(\mathbf{l}\,.\,d\mathbf{S})\, dt d\omega d\nu.$$

On the electromagnetic theory this description assumes that we have averaged over phases in some way; and the form is consistent with a particle theory.

Now consider the Lorentz transformation

$$\begin{aligned} x' &= (x+qt)/(1-q^2/c^2)^{\frac{1}{2}} \\ y' &= y \\ z' &= z \\ t' &= \{t+(qx/c^2)\}/(1-q^2/c^2)^{\frac{1}{2}}, \end{aligned}$$

in which a particle at rest in (x,y,z,t) is moving with velocity $(q, 0, 0)$ in (x',y',z',t').

Since
$$ct - x\cos\theta - y\sin\theta = cn/\nu$$
becomes
$$ct' - x'\frac{\cos\theta + q/c}{1+q\cos\theta/c} - y'\frac{\sin\theta(1-q^2/c^2)^{\frac{1}{2}}}{1+q\cos\theta/c} = \frac{cn(1-q^2/c^2)^{\frac{1}{2}}}{\nu(1+q\cos\theta/c)}$$

we have for a ray with direction given by (θ, ϕ), (θ', ϕ'),

$$\begin{aligned} \phi' &= \phi \\ \cos\theta' &= (\cos\theta + q/c)/(1+q\cos\theta/c) \\ \sin\theta' &= \sin\theta(1-q^2/c^2)^{\frac{1}{2}}/(1+q\cos\theta/c) \\ \nu' &= \nu(1+q\cos\theta/c)/(1-q^2/c^2)^{\frac{1}{2}}, \end{aligned}$$

equations giving aberration and Doppler effect.* Notice that

$$(1+q\cos\theta/c)(1-q\cos\theta'/c) = (1-q^2/c^2).$$

Since $d\omega = \sin\theta\, d\theta d\phi = d\cos\theta\, d\phi$,

$$d\omega' = d\omega(1-q^2/c^2)/(1+q\cos\theta/c)^2.$$

Further,

$$dt' = dt(1-q^2/c^2),\ (dx' = 0),\ \text{since } t = (t' - qx'/c^2)/(1-q^2/c^2)^{\frac{1}{2}}.$$

* Cf. Jeans, loc. cit. 2.

The number of quanta of the type considered passing through area dS perpendicular to the x-direction and moving so that $dx' = 0$ is proportional to

$$(I'_\nu/\nu')\, d\omega' dt' \cos\theta'\, dS d\nu',$$

and this must be the same as

$$(I_\nu/\nu)\, d\omega dt \cos\theta\, dS d\nu + (I_\nu/\nu)\, d\omega\, (q/c)\, dt dS d\nu,$$

the number which would have passed through dS, if it had moved so that $dx = 0$, together with those contained in a volume of base dS and height $q\,dt$, since $(I_\nu/c\nu)\, d\omega d\nu$ must be the number contained in unit volume, and since $q\, dt dS$ is the volume referred to x,y,z,t swept out by dS.

Hence $$I'_\nu = I_\nu(1+q\cos\theta/c)^3/(1-q^2/c^2)^{\frac{3}{2}}, \tag{4}$$

and $$I'_\nu\, d\omega' d\nu' = \{(1+q\cos\theta/c)^2/(1-q^2/c^2)\} I_\nu\, d\omega d\nu. \tag{4.1}$$

These results extend at once to the Lorentz transformation in any direction

$$\mathbf{r}' = \left\{\mathbf{r}(1-\mathbf{q}^2/c^2)^{\frac{1}{2}} + [1-(1-\mathbf{q}^2/c^2)^{\frac{1}{2}}]\frac{(\mathbf{q}\,.\,\mathbf{r})}{\mathbf{q}^2}\mathbf{q} + \mathbf{q}\frac{t}{c}\right\} \Big/ (1-\mathbf{q}^2/c^2)^{\frac{1}{2}}$$

$$t' = \{t + (\mathbf{q}\,.\,\mathbf{r})/c^2\}/(1-\mathbf{q}^2/c^2)^{\frac{1}{2}},$$

where $\mathbf{r} = (x, y, z)$.

The transformation-formula for direction-cosines of a light-track is

$$\mathbf{l}' = \left\{\mathbf{l}(1-\mathbf{q}^2/c^2)^{\frac{1}{2}} + [1-(1-\mathbf{q}^2/c^2)^{\frac{1}{2}}]\frac{(\mathbf{l}\,.\,\mathbf{q})}{\mathbf{q}^2}\mathbf{q} + \mathbf{q}/c\right\} \Big/ \{1+(\mathbf{l}\,.\,\mathbf{q})/c\} \tag{5}$$

while the factor $(1+q\cos\theta/c)/(1-q^2/c^2)^{\frac{1}{2}}$ becomes

$$\{1+(\mathbf{l}\,.\,\mathbf{q})/c\}/(1-\mathbf{q}^2/c^2)^{\frac{1}{2}}.$$

It is clear from the formula of transformation of I_ν that there is no inconsistency between the particle-description of radiation and restricted relativity.

Schwarzschild's equation for the transmission of radiation may be written

$$\frac{1}{c}\dot{I}_\nu + (\mathbf{l}\,.\,\nabla)I_\nu = -\rho k_\nu I_\nu + \rho\epsilon_\nu. \tag{6}$$

Generally, if it were necessary to take account explicitly of scattering and fluorescence, we should also have a term of the form

$$\iint \rho\epsilon(\nu,\nu',\mathbf{l}\,.\,\mathbf{l}')I_{\nu'}(\mathbf{l}')\, d\nu' d\omega'$$

on the right-hand side, and this could be treated in the same way.

Now

$$\frac{1}{c}\frac{\partial}{\partial t'}+(\mathbf{l}'.\nabla')=[(1-\mathbf{q}^2/c^2)^{\frac{1}{2}}/\{1+(\mathbf{l}.\mathbf{q})/c\}]\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}.\nabla)\right),$$

so that

$$-(\rho k_\nu)'I'_\nu+(\rho\epsilon_\nu)'=[\{1+(\mathbf{l}.\mathbf{q})/c\}^2/(1-\mathbf{q}^2/c^2)](-\rho k_\nu I_\nu+\rho\epsilon_\nu)$$

and

$$(\rho k_\nu)'=\rho k_\nu(1-\mathbf{q}^2/c^2)^{\frac{1}{2}}/\{1+(\mathbf{l}.\mathbf{q})/c\}$$

$$(\rho\epsilon_\nu)'=\rho\epsilon_\nu\{1+(\mathbf{l}.\mathbf{q})/c\}^2/(1-\mathbf{q}^2/c^2).$$

If, therefore, the values of ρk_ν and $\rho\epsilon_\nu$, referred to a coordinate-system in which $\mathbf{v}$ is zero at the point considered, are $\rho_0 k_{\nu 0}$ and $\rho_0\epsilon_{\nu 0}$, equation (6) becomes

$$\frac{1}{c}\dot{I}_\nu+(\mathbf{l}.\nabla)I_\nu=-\rho_0 k_{\nu 0}[\{1-(\mathbf{v}.\mathbf{l})/c\}/(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}]I_\nu+ \\ +\rho_0\epsilon_{\nu 0}(1-\mathbf{v}^2/c^2)/\{1-(\mathbf{v}.\mathbf{l})/c\}^2. \quad (6.1)$$

So long as we neglect very small terms depending on the departure of the velocity-distribution at a point of the matter from the Maxwellian, we may suppose $k_{\nu 0}$ and $\epsilon_{\nu 0}$ to have their equilibrium values.

4. The radiation stress-energy tensor. The rates of absorption of energy and momentum by the matter from the radiation are

$$\iint\{\rho k_\nu I_\nu-\rho\epsilon_\nu\}\,d\nu d\omega,\qquad \frac{1}{c}\iint\{\rho k_\nu I_\nu-\rho\epsilon_\nu\}\mathbf{l}\,d\nu d\omega,$$

and it is verified at once that

$$\left(\frac{1}{c^2}\iint\{\rho k_\nu I_\nu-\rho\epsilon_\nu\}\,d\nu d\omega,\qquad \frac{1}{c}\iint\{\rho k_\nu I_\nu-\rho\epsilon_\nu\}\mathbf{l}\,d\nu d\omega\right)$$

transforms as a contravariant four-vector.

Likewise

$$\begin{Bmatrix} \dfrac{1}{c^3}\displaystyle\iint I_\nu\,d\nu d\omega, & \dfrac{1}{c^2}\displaystyle\iint I_\nu\mathbf{l}\,d\nu d\omega \\ \dfrac{1}{c^2}\displaystyle\iint I_\nu\mathbf{l}\,d\nu d\omega, & \dfrac{1}{c}\displaystyle\iint I_\nu\mathbf{l}\mathbf{l}\,d\nu d\omega \end{Bmatrix}$$

transforms as a contravariant tensor, and by (6),

$$\frac{\partial}{\partial t}\frac{1}{c^3}\iint I_\nu\,d\nu d\omega+\nabla.\frac{1}{c^2}\iint I_\nu\mathbf{l}\,d\nu d\omega=\frac{1}{c^2}\iint\{\rho\epsilon_\nu-\rho k_\nu I_\nu\}\,d\nu d\omega$$

$$\frac{\partial}{\partial t}\frac{1}{c^2}\iint I_\nu\mathbf{l}\,d\nu d\omega+\nabla.\frac{1}{c}\iint I_\nu\mathbf{l}\mathbf{l}\,d\nu d\omega=\frac{1}{c}\iint\{\rho\epsilon_\nu-\rho k_\nu I_\nu\}\mathbf{l}\,d\nu d\omega.$$

Thus this tensor is the radiation stress-energy tensor and either side of these last equations may be added to the left-hand sides of equations (2.2) to replace the radiation terms in them.

Moreover, if one diagonal term of a symmetrical tensor is known in all coordinate-systems, the tensor is determined, so we need only actually calculate $\frac{1}{c^3}\iint I_\nu \, d\nu d\omega$ and can infer the other components from it.

To obtain I_ν, we solve (6.1) by successive approximations, obtaining

$$I_\nu = \frac{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^3}\frac{\epsilon_{\nu 0}}{k_{\nu 0}} -$$

$$- \frac{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}}\frac{1}{\rho_0 k_{\nu 0}}\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\frac{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^3}\frac{\epsilon_{\nu 0}}{k_{\nu 0}}+\dots,$$

where except under conditions of rapid variation* the two terms written down give a very good approximation.

The largest terms in $\frac{1}{c^3}\iint I_\nu \, d\nu d\omega$ are therefore given by

$$\frac{1}{c^3}\iint \frac{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^3}\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\, d\nu d\omega,$$

which is conveniently evaluated in terms of ν_0 and $\mathbf{l}_0$ as

$$\frac{1}{c_3}\iint \frac{\{1+(\mathbf{v}\,.\,\mathbf{l}_0)/c\}^2}{(1-\mathbf{v}^2/c^2)}\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\, d\nu_0 d\omega_0,$$

or, if

$$\int \frac{\epsilon_{\nu 0}}{k_{\nu 0}}\, d\nu_0 = \frac{c}{4\pi}E_0, \text{ since} \int (\mathbf{v}\,.\,\mathbf{l}_0)\, d\omega_0 = 0, \quad \int (\mathbf{v}\,.\,\mathbf{l}_0)^2\, d\omega_0 = \frac{4\pi}{3}\mathbf{v}^2,$$

$$\frac{1}{c^2}\frac{1+\mathbf{v}^2/3c^2}{1-\mathbf{v}^2/c^2}E_0 = \frac{1}{c^2}\left\{\frac{4}{3}\frac{E_0}{1-\mathbf{v}^2/c^2}-\frac{1}{3}E_0\right\},$$

from which the whole tensor has the form

$$\left\{\begin{matrix} \dfrac{4}{3c^2}\dfrac{E_0}{1-\mathbf{v}^2/c^2}-\dfrac{1}{3c^2}E_0 & \dfrac{4}{3c^2}\dfrac{E_0\mathbf{v}}{1-\mathbf{v}^2/c^2} \\ \dfrac{4}{3c^2}\dfrac{E_0\mathbf{v}}{1-\mathbf{v}^2/c^2} & \dfrac{4}{3c^2}\dfrac{E_0\mathbf{v}\mathbf{v}}{1-\mathbf{v}^2/c^2}+\dfrac{1}{3}E_0\mathbf{I} \end{matrix}\right\}, \tag{7}$$

or

$$E_0\left(\frac{4}{3c^2}V^\mu V^\nu+\frac{1}{3}g^{\mu\nu}\right). \tag{7.1}$$

* See Jeans, *Monthly Notices*, 86 (1926), 574.

The terms to be added to (2) can at once be written down,

$$\frac{\partial}{\partial x_\mu} E_0\left(\frac{4}{3c^2} V^\mu V^\nu + \frac{1}{3} g^{\mu\nu}\right),$$

and those to be added to (3) are

$$-g_{\nu\sigma} V^\sigma \frac{\partial}{\partial x_\mu} E_0\left(\frac{4}{3c^2} V^\mu V^\nu + \frac{1}{3} g^{\mu\nu}\right) = \frac{4}{3}\frac{\partial}{\partial x_\mu}(E_0 V^\mu) - \frac{1}{3} V^\mu \frac{\partial}{\partial x_\mu} E_0$$

$$= \frac{DE_0}{Dt} - \frac{4}{3}\frac{E_0}{\rho_0}\frac{D\rho_0}{Dt}$$

on subtracting $\frac{4}{3}E_0/\rho_0$ times (1), so that equation (3) becomes, if terms of higher order are neglected,

$$\rho_0 \frac{D}{Dt} U + \frac{D}{Dt} E_0 - \left(\frac{p}{\rho_0} + \frac{4}{3}\frac{E_0}{\rho_0}\right)\frac{D\rho_0}{Dt} = T_0 \tag{3.1}$$

where T_0 now contains no terms arising from either the ordinary temperature-radiation or from the gravitational field.

To obtain the terms of next higher order, which include radiative transfer of heat and radiative viscosity, we proceed in the same way, making use of the formulae

$$\frac{1}{4\pi}\int (\mathbf{l}_0 \,.\, \mathbf{a})(\mathbf{l}_0 \,.\, \mathbf{b})\, d\omega_0 = \tfrac{1}{3}(\mathbf{a} \,.\, \mathbf{b}),$$

$$\frac{1}{4\pi}\int (\mathbf{l}_0 \,.\, \mathbf{a})(\mathbf{l}_0 \,.\, \mathbf{b})(\mathbf{l}_0 \,.\, \mathbf{c})(\mathbf{l}_0 \,.\, \mathbf{d})\, d\omega_0$$

$$= \tfrac{1}{15}\{(\mathbf{a} \,.\, \mathbf{b})(\mathbf{c} \,.\, \mathbf{d}) + (\mathbf{a} \,.\, \mathbf{c})(\mathbf{b} \,.\, \mathbf{d}) + (\mathbf{a} \,.\, \mathbf{d})(\mathbf{b} \,.\, \mathbf{c})\}.$$

For higher terms further formulae are required, which are exactly analogous to the above; the next coefficient is 1/3.5.7 and all possible partitions of the six constant vectors into pairs occur. The corresponding integrals with an odd number of products are zero.

The next term in I_ν is

$$-\frac{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}} \frac{1}{\rho_0 k_{\nu 0}}\left(\frac{1}{c}\frac{\partial}{\partial t} + (\mathbf{l}\,.\,\nabla)\right)\frac{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^3}\frac{\epsilon_{\nu 0}}{k_{\nu 0}}.$$

Supposing now that $\epsilon_{\nu 0}/k_{\nu 0}$ is a function of ν_0 and T only, so that

$$d\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right) = \frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right) d\mathrm{T} + \frac{\partial}{\partial \nu_0}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)\nu\, d\left(\frac{1-(\mathbf{v}\,.\,\mathbf{l})/c}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\right),$$

we obtain

$$-\frac{(1-\mathbf{v}^2/c^2)^2}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^4}\frac{1}{\rho_0 k_{\nu 0}}\frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathrm{T}+$$

$$+\left\{3\frac{\epsilon_{\nu 0}}{k_{\nu 0}}-\nu_0\frac{\partial}{\partial \nu_0}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)\right\}\frac{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^5}\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\left(\frac{1-(\mathbf{v}\,.\,\mathbf{l})/c}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\right),$$

or, if we assume

$$\frac{\epsilon_{\nu 0}}{k_{\nu 0}}=\nu^3 f\left(\frac{\nu}{\mathrm{T}}\right),$$

so that

$$3\frac{\epsilon_{\nu 0}}{k_{\nu 0}}-\nu_0\frac{\partial}{\partial \nu_0}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)=\mathrm{T}\frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right),$$

we have

$$-\frac{(1-\mathbf{v}^2/c^2)^2}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}^4}\frac{1}{\rho_0 k_{\nu 0}}\frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)\times$$

$$\times\left[\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathrm{T}-\frac{\mathrm{T}}{(1-\mathbf{v}^2/c^2)}\left\{\frac{\mathbf{v}}{c^2}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathbf{v}\right\}+\right.$$

$$\left.+\frac{\mathrm{T}}{\{1-(\mathbf{v}\,.\,\mathbf{l})/c\}}\left\{\frac{\mathbf{l}}{c}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathbf{v}\right\}\right].$$

If we write

$$\int\frac{1}{k_{\nu 0}}\frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)d\nu_0=\frac{1}{\kappa}\int\frac{\partial}{\partial \mathrm{T}}\left(\frac{\epsilon_{\nu 0}}{k_{\nu 0}}\right)d\nu_0,$$

so that κ is the coefficient of opacity and is in general a function of T and ρ_0,* the contribution to $\frac{1}{c^3}\int I_\nu\,d\nu d\omega=\frac{1}{c^3}\int I_\nu\frac{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}{(1+\mathbf{v}\,.\,\mathbf{l}_0/c)}d\nu_0 d\omega_0$,

is, since $\int\frac{\epsilon_{\nu 0}}{k_{\nu 0}}d\nu_0=\frac{c}{4\pi}E_0$,

$$-\frac{1}{c^2\kappa\rho_0}\frac{\partial E_0}{\partial \mathrm{T}}\frac{1}{4\pi}\int\frac{\{1+(\mathbf{v}\,.\,\mathbf{l}_0)/c\}^3}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}\left[\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathrm{T}+\right.$$

$$+\mathrm{T}\frac{\{1+(\mathbf{v}\,.\,\mathbf{l}_0)/c\}}{(1-\mathbf{v}^2/c^2)}\left\{\frac{\mathbf{l}}{c}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathbf{v}\right\}-$$

$$\left.-\frac{\mathrm{T}}{(1-\mathbf{v}^2/c^2)}\left\{\frac{\mathbf{v}}{c^2}\cdot\left(\frac{1}{c}\frac{\partial}{\partial t}+(\mathbf{l}\,.\,\nabla)\right)\mathbf{v}\right\}\right]d\omega_0.$$

If in this expression $\mathbf{l}$ is replaced by its value in terms of $\mathbf{l}_0$ and $\mathbf{v}$,

$$\frac{\mathbf{l}_0(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}+\{1-(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}\}\{(\mathbf{l}_0\,.\,\mathbf{v})/\mathbf{v}^2\}\mathbf{v}+\mathbf{v}/c}{1+(\mathbf{v}\,.\,\mathbf{l}_0)/c},$$

* Rosseland, *Monthly Notices*, 84, 525.

and the integrations are carried out, the result is

$$-\frac{1}{c^2\kappa\rho_0}\frac{\partial E_0}{\partial \mathrm{T}}\left[\frac{1}{c}\frac{1+\mathbf{v}^2/c^2}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}\frac{\partial \mathrm{T}}{\partial t}+\frac{1}{c}\frac{\frac{5}{3}+\frac{1}{3}\mathbf{v}^2/c^2}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}(\mathbf{v}\,.\,\nabla)\mathrm{T}+\right.$$
$$+\frac{\mathrm{T}}{c^3}\frac{1+\frac{1}{5}\mathbf{v}^2/c^2}{(1-\mathbf{v}^2/c^2)^{\frac{5}{2}}}\left(\mathbf{v}\,.\,\frac{\partial}{\partial t}\mathbf{v}\right)+\frac{\mathrm{T}}{3c}\frac{1+\frac{1}{5}\mathbf{v}^2/c^2}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}(\nabla\,.\,\mathbf{v})+$$
$$\left.+\frac{\mathrm{T}}{3c^3}\frac{\frac{17}{5}+\frac{1}{5}\mathbf{v}^2/c^2}{(1-\mathbf{v}^2/c^2)^{\frac{5}{2}}}\{\mathbf{v}\,.\,(\mathbf{v}\,.\,\nabla)\mathbf{v}\}\right]$$
$$=-\frac{1}{c^2\kappa\rho_0}\frac{\partial E_0}{\partial \mathrm{T}}\left[\frac{2/c}{(1-\mathbf{v}^2/c^2)}\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\times\right.$$
$$\times\left(\frac{\partial}{\partial t}+(\mathbf{v}\,.\,\nabla)\right)\mathrm{T}-\frac{1/3c}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\left(\frac{\partial}{\partial t}+(\mathbf{v}\,.\,\nabla)\right)\mathrm{T}-$$
$$-\frac{2/3c}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\frac{\partial}{\partial t}\mathrm{T}+\frac{2\mathrm{T}}{5c}\left(\frac{\partial}{\partial t}\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}+\nabla\,.\,\frac{\mathbf{v}}{(1-\mathbf{v}^2/c^2)^{\frac{3}{2}}}\right)-$$
$$\left.-\frac{\mathrm{T}}{15c}\left(\frac{\partial}{\partial t}\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}+\nabla\,.\,\frac{\mathbf{v}}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\right)-\frac{2\mathrm{T}}{15c}\frac{\partial}{\partial t}\frac{1}{(1-\mathbf{v}^2/c^2)^{\frac{1}{2}}}\right]$$

from which the whole tensor has the form

$$-\frac{1}{c^3\kappa\rho_0}\frac{\partial E_0}{\partial \mathrm{T}}\left[2V^\mu V^\nu V^\tau\frac{\partial \mathrm{T}}{\partial x_\tau}+\frac{1}{3}g^{\mu\nu}c^2V^\tau\frac{\partial \mathrm{T}}{\partial x_\tau}+\frac{1}{3}c^2V^\mu g^{\nu\tau}\frac{\partial \mathrm{T}}{\partial x_\tau}+\right.$$
$$+\frac{1}{3}c^2V^\nu g^{\mu\tau}\frac{\partial \mathrm{T}}{\partial x_\tau}+\frac{2}{5}\mathrm{T}\frac{\partial}{\partial x_\tau}(V^\mu V^\nu V^\tau)+\frac{1}{15}g^{\mu\nu}c^2\mathrm{T}\frac{\partial}{\partial x_\tau}V^\tau+$$
$$\left.+\frac{1}{15}c^2\mathrm{T}g^{\mu\tau}\frac{\partial}{\partial x_\tau}V^\nu+\frac{1}{15}c^2\mathrm{T}g^{\nu\tau}\frac{\partial}{\partial x_\tau}V^\mu\right]. \quad (8)$$

the largest terms of which are

$$-\frac{1}{c^3\kappa\rho_0}\frac{\partial E_0}{\partial \mathrm{T}}\left\{\begin{matrix}\frac{5}{3}\frac{D\mathrm{T}}{Dt}-\frac{2}{3}\frac{\partial \mathrm{T}}{\partial t}+\frac{\mathrm{T}}{3}(\nabla\,.\,\mathbf{v}) & \frac{1}{3}c^2\nabla\mathrm{T} \\ \frac{1}{3}c^2\nabla\mathrm{T} & \begin{matrix}\frac{1}{3}c^2\mathbf{I}\frac{D}{Dt}\mathrm{T}+\frac{1}{3}c^2\mathbf{v}\nabla\mathrm{T}+\\ +\frac{1}{3}c^2(\nabla\mathrm{T})\mathbf{v}+\\ +\frac{1}{15}c^2\left(-\frac{1}{\rho_0}\frac{D\rho_0}{Dt}\right)+\\ +\frac{1}{15}c^2\nabla\mathbf{v}+\frac{1}{15}c^2\widetilde{\nabla\mathbf{v}}\end{matrix}\end{matrix}\right\}. \quad (8.1)$$

The terms to be added to (2) and (3) follow immediately. In particular the principal term in (3) is

$$-\nabla\,.\,\frac{c}{3\kappa\rho_0}\nabla E_0, \tag{9}$$

giving

$$\rho_0\frac{D}{Dt}U+\frac{D}{Dt}E_0-\left(\frac{p}{\rho_0}+\frac{4}{3}\frac{E_0}{\rho_0}\right)\frac{D\rho_0}{Dt}-\nabla\,.\,\frac{c}{3\kappa\rho_0}\nabla E_0=T_0. \tag{3.2}$$

5. Comparison with previous results. Equation (3.2) above agrees with the result of Jeans as corrected by Vogt*; the result given by Milne† in which $\frac{D}{Dt}E_0$ is replaced by $\frac{\partial}{\partial t}E_0$ and $-\frac{4}{3}\frac{E_0}{\rho_0}\frac{D\rho_0}{Dt}$ by $-\frac{1}{3}\frac{E_0}{\rho_0}\frac{D\rho_0}{Dt}$ requires, neglecting small relativity corrections, the addition to the 'radiative conduction' terms (9) a term $(\mathbf{v}\,.\,\nabla)E_0+E_0(\nabla\,.\,\mathbf{v})$ or $(\nabla\,.\,\mathbf{v}E_0)$; i.e. Jeans's $\mathbf{F}$ must be $c(3\kappa\rho_0)^{-1}\nabla E_0$, (*a*), while Milne's $\mathbf{F}$ must be $\mathbf{v}E_0-c(3\kappa\rho_0)^{-1}\nabla E_0$, (*b*). The value of $\mathbf{F}$ given by Milne,‡ however, is $\frac{4}{3}\,\mathbf{v}E_0-c(3\kappa\rho_0)^{-1}\nabla E_0$, (*c*). In fact (*a*) is the rate of flow of invariant energy (energy measured in a coordinate-system moving with the matter) in a coordinate-system moving with the matter, (*b*) is the rate of flow of invariant energy in a fixed coordinate-system, while (*c*) is the rate of flow of coordinate energy (the energy component of the energy-momentum vector) in a fixed coordinate-system.

The terms in the stress in (8.1) depending on the temperature-gradient agree with those given by Milne and differ from those given by Jeans because, as in Milne, but not as in Jeans, T here denotes the material temperature.

As regards radiative viscosity, however, the state of affairs is different. The principal terms in the stress-tensor representing viscosity are

$$-\frac{1}{15c\kappa\rho_0}\mathrm{T}\frac{\partial E_0}{\partial \mathrm{T}}\left\{\begin{matrix} 3\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} & \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} & 3\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} & \frac{\partial w}{\partial y}+\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z}+\frac{\partial w}{\partial x} & \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} & 3\frac{\partial w}{\partial z}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} \end{matrix}\right\}$$

* Loc. cit.; see also Jeans, *Astronomy and Cosmogony*, p. 118.

† *Quart. J. of Math.* (Oxford), 1 (1930), 8.

‡ *Monthly Notices*, 89 (1929), 523, equation (28).

The cross-terms have twice the coefficients found by Jeans and Milne, and the diagonal terms twice the coefficients found by Milne, Jeans differing further on account of his different definition of T. The corresponding term in the energy-density is likewise doubled. The error in Jeans's calculations occurs in the equation before his approximate equation (7).* The term

$$2\frac{I}{c}(du\sin\theta+dv\sin\theta\cos\phi+dw\sin\theta\sin\phi)$$

should be doubled, the extra part arising from the change of coordinate-energy from one coordinate-system to the other. That in Milne's is the same; his equation (5)† is wrong and the term $2(\mathbf{i}\,.\,\mathbf{v})/c$ in (6) should be doubled; his equation (14) is not invariant for Lorentz transformations and is not equivalent to (6.1) above.

6. On the inclusion of generation of energy in the equations. If generation of radiant energy at the expense of the number of particles in the matter is to be included in the equations, a term representing the disappearance of particles must be included in (1); as well as corresponding terms in (2), say $-\rho_0 G/c^2$ and $-\rho_{00} G V^\mu/c^2$ on the right-hand sides of (1) and (2). Terms representing the absorption of the radiation produced would have to be included also. This radiation, however, would not be part of the material temperature-radiation considered above, but would be treated separately by making assumptions suited to the physical theory of generation on which it was based. For instance, it might be assumed that G represented a property of matter independent of temperature and pressure and irreversibly producing high-frequency radiation absorbed by the matter at a definite rate $\rho_{00}K$. The above method could be applied to find the corresponding terms in the equations, $\int \rho_0\epsilon_{\nu 0}\,dv_0$ being replaced by $\rho_{00}G/4\pi$; and the first-order terms to be added to (2) would be

$$\frac{\partial}{\partial x_\mu}\frac{1}{c}\frac{G}{K}\left(\frac{4}{3c^2}V^\mu V^\nu+\frac{1}{3}g^{\mu\nu}\right)$$

and to (3), in addition to the corresponding terms,

$$-\frac{1}{c^2}\left(c^2\rho_{00}+p+\frac{4}{3}E_0+\frac{4}{3}\frac{1}{c}\frac{G}{K}\right)G$$

on account of the extra terms in (1). If we keep in the largest of

* *Monthly Notices*, 86 (1926), 446. † Ibid. 89 (1928), 520.

these terms only and insert the corresponding conduction-term, (3) becomes

$$\rho_0 \frac{D}{Dt} U + \frac{D}{Dt}\left(E_0 + \frac{1}{c}\frac{G}{K}\right) - \left(\frac{p}{\rho_0} + \frac{4}{3}\frac{E_0}{\rho_0} + \frac{4}{3}\frac{1}{c\rho_0}\frac{G}{K}\right)\frac{D\rho_0}{Dt}$$
$$= \rho_{00} G + \nabla \cdot \left(\frac{1}{3c\kappa\rho_0}\nabla E_0 + \frac{1}{3cK\rho_0}\nabla \frac{1}{c}\frac{G}{K}\right) \qquad (3.3)^*$$

In general, however, scattering should probably be taken explicitly into account.

7. **Summary.** The equations of transfer of radiation in a fluid in motion are obtained in a form including terms of all orders in the ratio of the velocity of motion to the velocity of light, but the calculations are carried only to the first-order terms, comprising radiative transfer of energy and momentum, in the coefficient of opacity. The results differ somewhat from those obtained previously by others. In particular the radiative viscosity is shown to be double that found by Jeans and Milne.

* This equation is, of course, essentially identical with either that of Jeans and Vogt or that of Milne, provided that the conduction-terms are written correctly.

CONVERGENCE AND SUMMABILITY CRITERIA FOR FOURIER SERIES†

By J. J. GERGEN

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PART I

1. Introduction. The first object in Part I of this paper is to establish the generalization, stated in Theorem I below, of the Lebesgue‡ criterion for the convergence of a Fourier series. The second object is to discuss the relation of this test to the six commonly recognized criteria and certain of their generalizations. It is found that in this form the Lebesgue test strictly includes the others under consideration. Incidentally, it will be proved that the continuity condition in one of the alternative forms of Lebesgue's original theorem is redundant.

In Part II we extend Theorem I in two directions, obtaining in Theorem II a somewhat more general convergence criterion and also a criterion for Cesàro summability§ of order ρ. Theorem I is contained in Theorem II; but it seems best, even at the expense of repetition, to give a separate proof. Numerous details, which tend to obscure the arguments and which do not occur in the proof of I, arise in that of II.

† The author is indebted to Professor Hardy, not only for his encouragement and his interest in this work, but also for suggesting, throughout the preparation for publication, several possible simplifications in the proofs and generalizations of the material.

‡ Lebesgue, **13**.

§ The series $\sum_{n=0}^{\infty} u_n$ is summable, to sum s, by Cesàro means of order ρ, where $\rho > -1$, if

$$\lim_{n\to\infty}\left[\frac{1}{A_n^{(\rho)}}\sum_{m=0}^{n} A_{n-m}^{(\rho)} u_m\right] = u,$$

where $A_0^{(\rho)} = 1$, and $A_n^{(\rho)} = \dfrac{(n+\rho)(n-1+\rho)\dots(1+\rho)}{n!}$ for $n > 0$. Summability of order zero is of course equivalent to convergence, and summability of order ρ implies summability of order ρ_1, to the same sum, if $\rho_1 > \rho$.

To simplify the writing, we shall suppose once and for all that the Fourier series

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nt$$

in question is that of an even function $f(t)$ which is integrable in the Lebesgue sense and periodic with period 2π. Moreover, we shall confine our attention to the behaviour of the series at the origin, that is, to the behaviour of the series

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n. \tag{1.1}$$

We write $$\phi_0(t) = \phi(t) = f(t) - s,$$

and $$\phi_r(t) = \frac{1}{\Gamma(r)} \int_0^t (t-u)^{r-1} \phi(u)\, du$$

for $r > 0$, where s is a number whose value will be assigned when the occasion arises. We denote by (C_r), (C_1^*), and (C) the conditions†

(C_r) $$\phi_r(x) = o(x^r),$$

(C_1^*) $$\phi_1^*(x) = \int_0^x |\phi(t)|\, dt = o(x),$$

(C) *the series* (1.1) *is summable, to sum s, by some Cesàro means.*

In order that (C) hold, it is necessary and sufficient, as Hardy and Littlewood‡ have shown, that (C_r) hold for some positive integer r.

The conditions sufficient for the convergence of the series (1.1), we recall, are then in

(D) *Dini's test:*§ $\phi(t)/t$ *is integrable on the interval* $(0, \pi)$;

(J) *Jordan's test:*‖ $f(t)$ *is of bounded variation on some interval* $(0, \xi)$, *where* $0 < \xi \leqslant \pi$;

(V) *de la Vallée-Poussin's test:*†† *the mean value*

$$F(t) = \frac{1}{t} \int_0^t f(u)\, du$$

is of bounded variation on $(0, \xi)$;

† The variable x is always positive, and the symbols o, O always refer to the limiting process $x \to +0$.

‡ Hardy and Littlewood, **5**.

§ Dini, **3**.

‖ Jordan, **11**.

†† de la Vallée-Poussin, **20**.

(Y) *Young's test*:† (C_0) *and*

$$\text{(Y}'\text{)} \qquad \int_0^x |d\{tf(t)\}| = O(x),$$

where the integral denotes the total variation of $tf(t)$ *in the interval* $(0, x)$;

(Y_P) *Pollard's generalization of* (Y):‡ (C_1) *and* (Y′);

(Y_{HL}) *Hardy's and Littlewood's generalization of* (Y):§ (C) *and* (Y′);

(HL) *Hardy's and Littlewood's test*:‖ (C) *and*

$$\text{(HL}'\text{)} \qquad \int_0^\xi |\Delta_x f(t)|^p \, dt = O(x)$$

for some $p \geqq 1$, *where* $\Delta_x f(t) = f(t+x) - f(t)$;

(L_1) *Lebesgue's test*: (C_1^*) *and*

$$(L_1') \qquad \int_x^\xi \frac{|\Delta_x f(t)|}{t} \, dt = o(1);$$

(L_2) *Lebesgue's alternative to* (L_1): (C_1^*) *and*

$$(L_2') \qquad \int_x^\xi \left| \Delta_x \left\{ \frac{\phi(t)}{t} \right\} \right| dt = o(1);$$

(L_P) *Pollard's generalization of* (L_2):†† (C_1) *and*

$$(L_P') \qquad \lim_{k \to \infty} \overline{\lim_{x \to +0}} \int_{kx}^\xi \left| \Delta_x \left\{ \frac{\phi(t)}{t} \right\} \right| dt = 0.$$

In (J) *the sum of the series is* $f(+0)$, *in* (V) *it is* $F(+0)$, *and in the remaining tests* s.

The relations between these tests have been considered by several authors.‡‡ The following diagram indicates, with reservations enu-

† Young, **22**. In **23** Young states that (Y′) and (C_r), for any r, are sufficient for convergence. No proof for the case $r = 1$ seems, however, to have appeared until Pollard's paper **14**, and no proof for the general case until Hardy's and Littlewood's paper, **6**.

‡ Pollard, **14**.

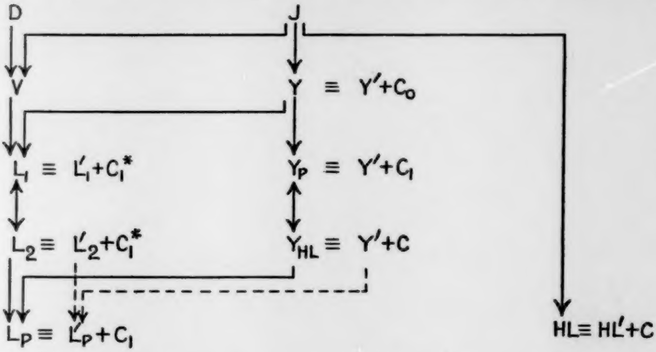
§ Hardy and Littlewood, **6**. See also Littauer, **25**. Littauer proves (Y_{HL}) by a method essentially different from that used by Hardy and Littlewood.

‖ Hardy and Littlewood, **7**. The theorem was originally proved with $\xi = \pi$. It is, however, a simple matter to deduce from this the test stated above.

†† Pollard, **14**.

‡‡ In the notation of the diagram, Pollard shows in **14** that (Y′) implies

merated below, all the implications which are known, so as far the author is aware. In this diagram, the notation $L_1 \equiv L_1' + C_1^*$ signifies the equivalence of the condition represented by the letter on the left to the two conditions represented by the letters on the right. A directed line running from one letter to another indicates that the condition represented by the former implies that represented by the latter. The full lines show the relations between the tests, and



the broken lines the relations between the characteristic parts of the tests. Identical conditions are not connected, nor are the relations

$$(C_0) \supset (C_1^*) \supset (C_1) \supset (C)\dagger$$

indicated. In the relations involving (J), s is understood to have the value $f(+0)$, and, in those involving (V), the value $F(+0)$.

The phrase 'characteristic parts of the tests' used above demands

(L_P'), Hardy and Littlewood show in **7** that (J) implies (HL), and in **5** that (Y') implies $$(C) \equiv (C_1),$$ so that $$(Y_P) \equiv (Y_{HL}) \supset (L_P).$$ It is trivial that $$(Y) \supset (Y_P), \quad (L_2') \supset (L_P'), \quad (L_2) \supset (L_P).$$ The remaining relations, together with references to the previous literature, are given by Hardy in **4**. He considers only the case in which $f(t)$ is continuous at the origin; but it is a simple matter to deduce with his methods the implications shown.

In a summary of this kind, it should perhaps be pointed out again that, for reasons often stated, the importance of any test is not less because the test is a particular case of, or equivalent to, another. In particular, the test (Y_{HL}) deserves comment in this respect. The degree of generality of (C) over (C_1) certainly places (Y_{HL}) in a different class from (L_P), and gives it, because of the economy in continuity conditions, an importance which is quite independent of its relation to (Y_P).

† Here '$\supset$', as usual, implies 'implies'.

a word of explanation. Most of the tests contain two conditions, one of which is continuity, or some generalization of continuity such as (C_1^*), (C_1), or (C), and the other a condition more individual to the test in question. We refer to the two conditions generally as 'the continuity condition' and 'the characteristic condition': thus in (Y) the continuity condition is (C_0) and the characteristic condition is (Y').

An examination of the diagram reveals that several questions pertaining both to the continuity conditions and to the characteristic conditions in the various forms of the Lebesgue test are unanswered. In particular, one may ask whether (C_1^*) may be replaced by (C_1) in (L_1), or, more generally, (C_1) by (C) and (L'_P) by

$$(L'_R) \qquad \lim_{k\to\infty} \overline{\lim_{x\to+0}} \int_{kx}^{\xi} \frac{|\Delta_x f(t)|}{t}\, dt = 0$$

in (L_P).† The answers to these questions are found to be in the affirmative in the first part of

THEOREM I. *If* (L'_R) *and* (C) *hold, the series* (1.1) *converges, to sum s. Furthermore, if* (L'_R) *holds, then*

$$(C) \equiv (C_1). \tag{1.2}$$

The second part of this theorem is of course a generalization of Hardy's and Littlewood's‡ result that (1.2) is a consequence of either (Y') or (HL').

To show that the conditions of the theorem, viz.

$$(L_R) \equiv (L'_R)+(C),$$

are implied by those of the tests enumerated, we need evidently only prove that

$$(HL') \supset (L'_R) \quad \text{and} \quad (L'_P) \supset (L'_R).$$

The first of these implications has, however, already been observed by Hardy and Littlewood;§ the second we shall prove in section 4.‖ This proof is effected by first showing that (L'_P) implies

$$\int_0^x |\phi(t)|\, dt = O(x); \tag{1.3}$$

† In **7** Hardy and Littlewood discuss these questions to some extent with reference to their test (HL). They do not arrive at any definite conclusion but express doubt that any of these changes can be made.

‡ Hardy and Littlewood, **6** and **7**.

§ Hardy and Littlewood, **7**. Their proof is given also in section 6 of this paper.

‖ It might be noted that (L'_R) does not imply (L'_P). Witness the example: $f(t) = \log t$.

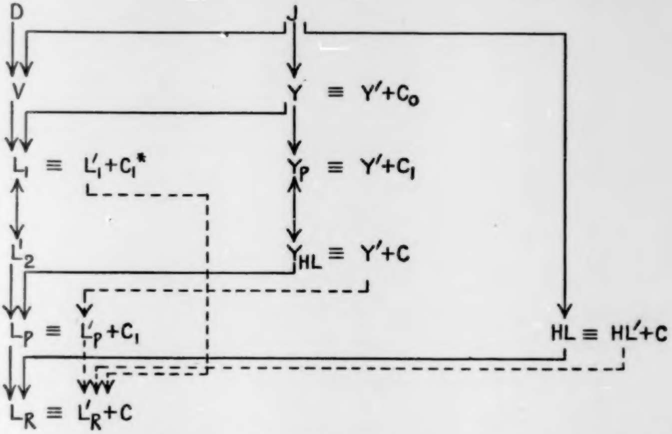
and it is by a slight change in this argument that we deduce in section 5 that

$$(\mathrm{L}_2') \supset (\mathrm{C}_1^*).$$

Incorporating these results, and also the trivial relation

$$(\mathrm{L}_1') \supset (\mathrm{L}_\mathrm{R}'),$$

into the previous diagram, we obtain the one following. It is interesting to observe that not only is (L_R) a consequence of the conditions



of every other test, but so also is (L_R') a consequence of every other characteristic condition.

Perhaps a further word should be added in regard to the proof of I. First, the proof is of an essentially different character from that of Hardy's and Littlewood's proof of (HL), in that it involves neither the theory of functions of a complex variable nor negative orders of summability. Secondly, it is in part of the same character as the usual proof of (L_1). The lemmas of sections 2.2, 2.3, and 2.4, in which it is shown that (L_R') and (C) imply (C_1), are of course without analogue in the proof of Lebesgue's theorem, but the proof that (L_R') and (C_1) imply convergence (see sections 2.5, 2.6, and 3) follows the latter in principle, if not in detail, except at one or two points.

2. Lemmas for Theorem I. 2.1. The proof of Theorem I rests on the following lemmas, some of which are known. In these lemmas and throughout the rest of the paper, $\Delta_x^{(m)}\Phi(t)$ represents the mth difference

$$\Delta_x^{(m)}\Phi(t) = \sum_{\nu=0}^{m} (-1)^{m+\nu}\binom{m}{\nu}\Phi(t+\nu x)$$

of the function Φ, and

$$\eta(x,k)=\int_{kx}^{\xi}\frac{|\Delta_x f(t)|}{t}\,dt=\int_{kx}^{\xi}\frac{|\Delta_x \phi(t)|}{t}\,dt.$$

We always suppose $k>1$, and write

$$\alpha(x,k)=\bar{o}\,(1),$$

for any function α of x and k, when, and only when,

$$\lim_{k\to\infty}\ \overline{\lim_{x\to+0}}\,|\alpha(x,k)|=0.$$

2.2. LEMMA 1. *If* (C) *holds, then* (C_r) *holds for some integer* r.

This is one of the theorems of Hardy and Littlewood cited in the Introduction.

2.3. LEMMA 2. *If* $kx\leqslant v$, *then*

$$\int_{kx}^{v}|\Delta_x^{(r+1)}\phi_r(t)|\,dt\leqslant(v+rx)x^r\int_{kx}^{v+rx}\frac{|\Delta_x\phi(t)|}{t}\,dt \tag{2.31}$$

for every integer $r\geqslant 0$.

This relation is plainly true when r is zero. Moreover, assuming it true when r is replaced by $r-1$, we have

$$\begin{aligned}
\int_{kx}^{v}|\Delta_x^{(r+1)}\phi_r(t)|\,dt &= \int_{kx}^{v}\left|\sum_{\nu=0}^{r+1}(-1)^\nu\binom{r+1}{\nu}\int_0^{t+\nu x}\phi_{r-1}(u)\,du\right|dt\\
&= \int_{kx}^{v}\left|\sum_{\nu=0}^{v}(-1)^\nu\binom{r}{\nu}\int_{t+\nu x}^{t+(\nu+1)x}\phi_{r-1}(u)\,du\right|dt\\
&= \int_{kx}^{v}\left|\int_t^{t+x}\left\{\sum_{\nu=0}^{r}(-1)^\nu\binom{r}{\nu}\phi_{r-1}(u+\nu x)\right\}du\right|dt\\
&\leqslant \int_{kx}^{v}dt\int_t^{t+x}|\Delta_x^{(r)}\phi_{r-1}(u)|\,du\\
&= \int_{kx}^{(k+1)x}(u-kx)|\Delta_x^{(r)}\phi_{r-1}(u)|\,du+x\int_{(k+1)x}^{v}|\Delta_x^{(r)}\phi_{r-1}(u)|\,du+\\
&\qquad +\int_v^{v+x}(v-u+x)|\Delta_x^{(r)}\phi_{r-1}(u)|\,du\\
&\leqslant x\int_{kx}^{v+x}|\Delta_x^{(r)}\phi_{r-1}(u)|\,du\leqslant(v+rx)x^r\int_{kx}^{v+rx}\frac{|\Delta_x\phi(u)|}{u}\,du.
\end{aligned}$$

The lemma then follows by induction.

2.4. LEMMA 3. *If* (L'_R) *holds, and* (C_r) *holds for some integer* $r \geqslant 0$, *then* $\phi_1(x) = o(x)$.

It is sufficient to prove that if $r \geqslant 1$ and

$$\phi_{r+1}(x) = o(x^{r+1}), \tag{2.41}$$

then $\phi_r(x) = o(x^r)$.

We first observe that

$$hx\,\phi_r(x) = \Phi(x,k) + \int_0^{hx} dt \int_{kt}^{x-rt} \Delta_t^{(r)}\phi_{r-1}(u)\,du, \tag{2.42}$$

where $h = (k+r)^{-1}$ and $\Phi(x,k)$ is of the form

$$\Phi(x,k) = \psi\,\phi_{r+1}(hkx) + \psi\,\phi_{r+1}\{h(k+1)x\} + \ldots + \psi\phi_{r+1}(x),$$

the ψ's being independent of x. In fact,

$$\int_0^{hx} dt \int_{kt}^{x-rt} \Delta_t^{(r)}\phi_{r-1}(u)\,du = \int_0^{hx} dt \int_{kt}^{x-rt} \left\{\sum_{\nu=0}^{r} (-1)^{r+\nu}\binom{r}{\nu}\phi_{r-1}(u+\nu t)\right\} du$$

$$= \sum_{\nu=0}^{r} (-1)^{r+\nu}\binom{r}{\nu} \int_0^{hx} dt \int_{(k+\nu)t}^{x+(\nu-r)t} \phi_{r-1}(u)\,du$$

$$= \int_0^{hx} \left\{\phi_r(x) - \phi_r\left(\frac{t}{h}\right)\right\} dt + \sum_{\nu=0}^{r-1} \psi \int_0^{hx} [\phi_r\{x+(\nu-r)t\} - \phi_r\{(k+\nu)t\}]\,dt$$

$$= hx\,\phi_r(x) + \psi\,\phi_{r+1}(x) + \sum_{\nu=0}^{r-1} \psi \int_{h(k+\nu)x}^{x} \phi_r(t)\,dt + \sum_{\nu=0}^{r-1} \psi \int_0^{h(k+\nu)x} \phi_r(t)\,dt$$

$$= hx\,\phi_r(x) + \sum_{\nu=0}^{r} \psi\,\phi_{r+1}\{h(k+\nu)x\} = hx\,\phi_r(x) - \Phi(x,k).$$

Now, by Lemma 2,

$$\left|\int_0^{hx} dt \int_{kt}^{x-rt} \Delta_t^{(r)}\phi_{r-1}(u)\,du\right| \leqslant \int_0^{hx} (x-t)t^{r-1}\,dt \int_{kt}^{x-t} \frac{|\Delta_t\phi(u)|}{u}\,du,$$

since kt does not exceed $x-rt$ in the integral $\int_{kt}^{x-rt}$. Hence, by (2.42),

$$\frac{|\phi_r(x)|}{x^r} \leqslant \frac{|\Phi(x,k)|}{hx^{r+1}} + \frac{h^{r-2}}{x}\int_0^{hx} \eta(t,k)\,dt$$

if $x \leqslant \xi$. But if (2.41) holds, the first function on the right is evidently $o(1)$ for every k, while if (L'_R) holds, the second is $\bar{o}(1)$. Accordingly,

$$|\phi_r(x)|/x^r = \bar{o}(1);$$

and thus, since $\phi_r(x)/x^r$ is independent of k, $\phi_r(x) = o(x^r)$. This proves the lemma.

2.5. LEMMA 4. *If* (C_1) *holds, then*

$$\int_a^{a+kx} \frac{\phi(t)}{t} \sin\frac{\pi t}{x}\, dt = o(1)$$

for every $a \geqslant 0$ *and* k.†

If $a > 0$, the lemma follows from the continuity of the integral $\int|\phi(t)|\, dt$. If $a = 0$, it becomes evident after an integration by parts. In fact,

$$\left|\int_0^{kx} \frac{\phi(t)}{t} \sin\frac{\pi t}{x}\, dt\right| = \left|\frac{\phi_1(kx)}{kx}\sin \pi k - \frac{\pi}{x}\int_0^{kx} \frac{\phi_1(t)}{t}\cos\frac{\pi t}{x}\, dt + \int_0^{kx}\frac{\phi_1(t)}{t^2}\sin\frac{\pi t}{x}\, dt\right|$$

$$\leqslant \frac{|\phi_1(kx)|}{kx} + \frac{2\pi}{x}\int_0^{kx} \frac{|\phi_1(t)|}{t}\, dt = o(1).$$

2.6. LEMMA 5. *If* $\phi_1(x) = O(x)$, *then*

$$\beta(x, k) = x^2 \int_{kx}^{\xi} \frac{\phi(t+x)}{t(t+x)(t+2x)} \sin\frac{\pi t}{x}\, dt = \bar{o}(1).$$

We have

$$0 < g(x, t) = \frac{x^2}{t(t+x)(t+2x)} < \frac{x^2}{t^2(t+x)},$$

$$0 < g_1(x, t) = -\frac{\partial g}{\partial t} < \frac{3x^2}{t^3(t+x)}$$

for $t > 0$. Hence

$$|\beta(x, k)| = \left|\left[\phi_1(t+x)g(x, t)\sin\frac{\pi t}{x}\right]_{kx}^{\xi} - \frac{\pi}{x}\int_{kx}^{\xi} \phi_1(t+x)g(x, t)\cos\frac{\pi t}{x}\, dt + \right.$$

$$\left. + \int_{kx}^{\xi} \phi_1(t+x)g_1(x, t)\sin\frac{\pi t}{x}\, dt\right|$$

$$\leqslant \frac{Mx^2}{\xi^2} + \frac{M}{k^2} + 4\pi Mx\int_{kx}^{\xi}\frac{dt}{t^2} \leqslant \frac{Mx^2}{\xi^2} + \frac{5\pi M}{k},$$

for $kx \leqslant \xi$, where M is an upper bound of $|\phi_1(t)|/t$ on $(0, 2\xi)$. From this last inequality the lemma evidently follows.

† The proof of this lemma may be found in Pollard's paper, **14**.

3. Proof of Theorem I. If (C) and (L'_R) hold, then (C_1) holds, by Lemmas 1 and 3. It suffices, then, for the justification of the complete theorem, to prove that (L'_R) and (C_1) imply the convergence of the series, to sum s.

Consider the integral

$$\sigma(x)=\int_0^\xi \frac{\phi(t)}{t}\sin\frac{\pi t}{x}\,dt$$

It is classical that the series converges to s if $\sigma(x)=o(1)$. To show that this is the case, we set

$$\alpha(x,k)=\left\{\int_0^{kx}+2\int_0^{(k+1)x}+\int_0^{(k+2)x}-2\int_\xi^{\xi+x}-\int_\xi^{\xi+2x}\right\}\frac{\phi(t)}{t}\sin\frac{\pi t}{x}\,dt,$$

and write

$$\begin{aligned}4\sigma(x)-\alpha(x,k)&=\left\{\int_{kx}^{\xi}+2\int_{(k+1)x}^{\xi+x}+\int_{(k+2)x}^{\xi+2x}\right\}\frac{\phi(t)}{t}\sin\frac{\pi t}{x}\,dt\\ &=2x^2\int_{kx}^{\xi}\frac{\phi(t+x)}{t(t+x)(t+2x)}\sin\frac{\pi t}{x}\,dt+\\ &\qquad+\int_{kx}^{\xi}\left\{\frac{\Delta_x\phi(t+x)}{t+2x}-\frac{\Delta_x\phi(t)}{t}\right\}\sin\frac{\pi t}{x}\,dt\\ &=2\beta(x,k)+\gamma(x,k), \text{ say.}\end{aligned}$$

Now if (C_1) holds,
$$\alpha(x,k)=o(1), \tag{3.1}$$
for every k, by Lemma 4, and
$$\beta(x,k)=\bar{o}(1), \tag{3.2}$$
by Lemma 5; while, if (L'_R) holds,
$$\gamma(x,k)=\bar{o}(1) \tag{3.3}$$
because of the continuity of the integral $\int|\phi(t)|\,dt$ and the fact that

$$|\gamma(x,k)|\leqslant 2\eta(x,k)+\frac{1}{\xi}\int_\xi^{\xi+2x}|\phi(t)|\,dt$$

for $kx\leqslant\xi$.

We conclude from (3.1), (3.2), and (3.3) that $\sigma(x)=o(1)$; which completes the proof.

4. Proof that (L'_P) implies (L'_R). We first show that (L'_P) implies (1.3). Denoting by $\mu(x,k)$ the least upper bound of

$$\int_{ky}^{\xi}\left|\Delta_y\left\{\frac{\phi(t)}{t}\right\}\right|\,dt$$

for a fixed k, for $0<y\leqslant x$, and, setting $x_\nu = x\{k/(k+1)\}^\nu$ for $\nu=0,1,\ldots$, we have

$$\int_0^{kx} |\phi(t)|\,dt = \sum_{\nu=0}^{\infty} \int_{kx_{\nu+1}}^{kx_\nu} |\phi(t)|\,dt \leqslant \sum_{\nu=0}^{\infty} kx_\nu \int_{kx_{\nu+1}}^{kx_\nu} \frac{|\phi(t)|}{t}\,dt$$

$$= k\sum_{\nu=0}^{\infty} x_\nu \left\{ \int_{kx_{\nu+1}}^{\xi} - \int_{kx_\nu}^{\xi+x_{\nu+1}} + \int_{\xi}^{\xi+x_{\nu+1}} \right\} \frac{|\phi(t)|}{t}\,dt$$

$$= k\sum_{\nu=0}^{\infty} x_\nu \left\{ \int_{kx_{\nu+1}}^{\xi} \frac{|\phi(t)|}{t}\,dt - \int_{kx_{\nu+1}}^{\xi} \frac{|\phi(t+x_{\nu+1})|}{t+x_{\nu+1}}\,dt + \int_{\xi}^{\xi+x_{\nu+1}} \frac{|\phi(t)|}{t}\,dt \right\}$$

$$\leqslant k\sum_{\nu=0}^{\infty} x_\nu \left\{ \mu(x,k) + \int_{\xi}^{\xi+x} \frac{|\phi(t)|}{t}\,dt \right\} = 2(k+1)\left\{ \mu(x,k) + \int_{\xi}^{\xi+x} \frac{|\phi(t)|}{t}\,dt \right\} \quad (4.1)$$

for $kx\leqslant\xi$. But if (L'_P) holds, $\mu(x,k)=O(1)$ for some k. Hence (1.3) holds.

Consider now $\eta(x,k)$. We have

$$\int_{kx}^{\xi} \frac{|\Delta_x f(t)|}{t}\,dt \leqslant \int_{kx}^{\xi} \left| \Delta_x \left\{ \frac{\phi(t)}{t} \right\} \right| dt + x\int_{kx}^{\xi} \frac{|\phi(t+x)|}{t(t+x)}\,dt$$

$$\leqslant \int_{kx}^{\xi} \left| \Delta_x \left\{ \frac{\phi(t)}{t} \right\} \right| dt + \left| \left[\frac{x}{t(t+x)} \int_0^{t+x} |\phi(u)|\,du \right]_{kx}^{\xi} \right| +$$

$$+ 2x\int_{kx}^{\xi} \frac{dt}{t^2(t+x)} \int_0^{t+x} |\phi(u)|\,du$$

$$\leqslant \int_{kx}^{\xi} \left| \Delta_x \left\{ \frac{\phi(t)}{t} \right\} \right| dt + \frac{Mx}{\xi} + \frac{3M}{k}$$

for $kx\leqslant\xi$, where M is an upper bound of $\dfrac{1}{x}\displaystyle\int_0^x |\phi(u)|\,du$ on the interval $(0, 2\xi)$. Hence, if (L'_P) and (1.3) hold, so also does (L'_R).

5. Proof that (L'_2) implies (C^*_1).

This is an immediate consequence of (4.1). If (L'_2) holds, we have

$$\mu(x,2) = o(1) \text{ and } \int_{\xi}^{\xi+x} \frac{|\phi(t)|}{t}\,dt = o(1).$$

Hence
$$\int_0^x |\phi(t)|\,dt = x\,o(1);$$
and this is (C^*_1).

PART II

6. A criterion for summability. We obtain the generalization of Theorem I, mentioned in the introduction, by replacing (L'_R) by the condition†

$$\eta_\rho^{(m)}(x,k) = x^\rho \int_{kx}^{\pi} \frac{|\Delta_x^{(m)} f(t)|}{t^{1+\rho}}\, dt = \bar{o}(1). \tag{6.1}$$

We prove

THEOREM II.‡ *If* $\rho > -1$, *and* (6.1) *and* (C) *hold, the series* (1.1) *is summable* (C, ρ),§ *to sum* s. *Furthermore, if* $-1 < \rho \leqslant 0$ *and* (6.1) *holds, then*

$$(C) \equiv (C_1),$$

while if $\rho > 0$ *and* (6.1) *holds,*

$$(C) \equiv (E),$$

where (E) *represents the two conditions*

$$\phi_1(x) = O(x), \tag{6.2}$$

$$\phi_2(x) = o(x^2).\| \tag{6.3}$$

† A generalization with mth differences of the Lebesgue criterion has been given by de la Vallée-Poussin. See **21**, p. 150.

‡ The condition (6.1) is of course equivalent to that obtained by replacing the upper limit π by any number $0 < \xi \leqslant \pi$, provided $\rho > 0$. This is no longer the case when $\rho < 0$; and it is, in fact, essential in the proof that the upper limit be π.

§ That is, summable by Cesàro means of order ρ.

‖ When $\rho > 0$ the conditions (6.1) and (C) do not imply (C_1). In fact, Hardy and Littlewood give in **10** an example of a function $\phi(t)$ for which

$$\int_0^x |\phi(t)|\, dt = O(x), \qquad \phi_1(x) \neq o(x), \qquad \phi_2(x) = o(x^2);$$

and we prove below that

$$\int_0^x |\phi(t)|\, dt = O(x)$$

implies (6.1) for every $\rho > 0$. However, it is worth noting in this connexion that (C) and

$$x^\rho \int_x^\pi \frac{|\Delta_x f(t)|}{t^{1+\rho}}\, dt = o(1)$$

imply (C_1) for every ρ, while (C) and (6.1), with $\rho > -1$, imply $(C_{1+\rho+\delta})$ for every $\delta > 0$. The first of these facts is a consequence of (8.26); and the second a consequence of the above theorem and Bosanquet's result (see **1**) that summability (C, ρ) implies $(C_{1+\rho+\delta})$.

The proof depends largely upon the following lemma.

LEMMA 6. *Suppose that* $-1 < \rho \leqslant 1$, *that* $S_n^{(\rho)}(t)$ *is the n-th Cesàro mean*

$$S_n^{(\rho)}(t) = \tfrac{1}{2} + \frac{1}{A_n^{(\rho)}} \sum_{\nu=1}^{n} A_{n-\nu}^{(\rho)} \cos \nu t$$

of order ρ *of the series* $\quad \tfrac{1}{2} + \sum_{\nu=1}^{\infty} \cos \nu t,$

that
$$T_n^{(\rho)}(t) = \frac{1}{A_n^{(\rho)}} \frac{\sin\{(n+\frac{1}{2}+\frac{1}{2}\rho)t - \frac{1}{2}\rho\pi\}}{(2 \sin \frac{1}{2}t)^{1+\rho}},$$

and that
$$R_n^{(\rho)}(t) = S_n^{(\rho)}(t) - T_n^{(\rho)}(\rho).$$

Then, if $0 < t \leqslant 3\pi/2$, *we have*

$$|S_n^{(\rho)}| < Mn, \qquad \left|\frac{dS_n^{(\rho)}}{dt}\right| < Mn^2, \tag{6.4), (6.5}$$

$$\left|\frac{d^2S_n^{(\rho)}}{dt^2}\right| < Mn^3, \qquad |R_n^{(\rho)}| < \frac{M}{nt^2}, \tag{6.6), (6.7}$$

$$\left|\frac{dR_n^{(\rho)}}{dt}\right| < \frac{M}{nt^3} + \frac{M}{n^2t^4}, \tag{6.8}$$

where the M's are independent of n and t.

The first three of these inequalities are trivial. The last but one, (6.7), was first obtained in the general case $-1 < \rho < 1$ by Kogbetliantz,† by using Cauchy's integral theorem. He shows that

$$R_n^{(\rho)}(t) = \frac{\sin \rho\pi}{2\pi A_n^{(\rho)}} \int_0^1 \frac{(1+u)u^{n+\rho}}{(1-u)^\rho (1-2u\cos t + u^2)}\, du;$$

and from this equation both (6.7) and (6.8) follow. The proof of (6.7) given here is substantially the same as Szegö's‡ proof, which we reproduce in order to obtain (6.8). We follow this method chiefly to eliminate, so far as possible, the dependence of Theorem II on the theory of functions of a complex variable. The whole proof of (6.7) and (6.8) rests on the formula $1 - e^{-it} = 2e^{i(\pi-t)/2} \sin \frac{1}{2}\theta$, where $i = \sqrt{(-1)}$, and on the elementary fact that, if $\rho < 1$, the branch of

† Kogbetliantz, **12**. For the formula in question see p. 277.

‡ Szegö, **19**. Szegö's arguments are of the same character as those previously used by Riesz, **16**, in proving that

$$|S_n^{(\rho)}| < Mn^{-\rho}t^{-(1+\rho)}$$

in the case $0 < \rho < 1$.

the function $(1-z)^{1-\rho}$ which is equal to $+1$ at the origin, is represented everywhere in the circle $|z| \leqslant 1$, with the exception of the point $z = 1$, by the convergent series

$$(1-z)^{1-\rho} = 1 + \sum_{\nu=1}^{\infty} A_\nu^{(\rho-2)} z^\nu, \tag{6.9}$$

in which $$A_\nu^{(\rho-2)} = \frac{(\nu+\rho-2)(\nu+\rho-3)\dots\rho(\rho-1)}{\nu!}.$$

Two particular cases of Theorem II are worth noting:

(HL$_1$) *If* (HL′) *holds, with* $\xi = \pi$, *and if* (C) *holds, then the series* (1.1) *is summable* $(C, -1/p+\delta)$, *to sum* s, *for every positive* δ. *Moreover, if* (HL′) *holds,*

$$(C) \equiv (C_1).$$

(HL$_2$) *If* $\int_0^x |\phi(t)|\, dt = O(x)$ *and* (C) *holds, then the series* (1.1) *is summable* (C, δ), *to sum* s, *for every positive* δ. *Moreover, if* $\int_0^x |\phi(t)|\, dt = O(x)$,

$$(C) \equiv (C_{1+\delta}).$$

Both of these theorems are due to Hardy and Littlewood;† and although their proofs as originally given are perhaps shorter and more illuminating than that of II, it is interesting that alternative proofs do exist.

That (HL$_1$) is a corollary of Theorem II is a consequence of the inequalities

$$x^{-1/p+\delta} \int_{kx}^{\pi} \frac{|\Delta_x f(t)|}{t^{1/q+\delta}}\, dt \leqslant x^{-1/p+\delta} \left(\int_{kx}^{\infty} \frac{dt}{t^{1+q\delta}} \right)^{1/q} \left(\int_0^{\pi} |\Delta_x f|^p\, dt \right)^{1/p}$$

$$< \frac{M}{k^\delta} \left(\frac{1}{x} \int_0^{\pi} |\Delta_x f|^p\, dt \right)^{1/p},$$

where $p^{-1}+q^{-1} = 1$ and M is independent of x and k. That (HL$_2$)

† See Hardy and Littlewood, **7**, for (HL$_1$), and **8** and **10** for (HL$_2$). They state (HL$_2$) in **8** and prove it in **10**. Pollard, **15**, proves, by a method different from that used by Hardy and Littlewood, the particular case of (HL$_2$), that (C_1) and $\int_0^x |\phi(t)|\, dt = O(x)$ imply summability (C, 1).

is also a corollary of Theorem II is a consequence of Lemma 9 below, and the fact that

$$x^\rho \int_{kx}^{\pi} \frac{|\Delta_x f(t)|}{t^{1+\rho}}\, dt < Mx^\rho \int_{kx}^{\pi} \frac{|f(t)|}{t^{1+\rho}} < Mx^\rho + Mx^\rho \int_{kx}^{\infty} \frac{dt}{t^{1+\rho}}$$

$$< Mx^\rho + Mk^{-\rho} = \bar{o}(1)$$

if $\rho > 0$ and $\int_0^x |\phi(t)|\, dt = O(x)$.

7. Proof of Lemma 6. The proof of (6.4), (6.5), and (6.6) is immediate. We have

$$|S_n^{(\rho)}| < \frac{1}{2} + \frac{M}{n^\rho} \sum_{\nu=1}^{n} (n-\nu)^\rho < Mn,$$

since

$$A_n^{(\rho)} \sim n^\rho / \Gamma(1+\rho); \tag{7.1}$$

and similar inequalities hold for $dS_n^{(\rho)}/dt$, $d^2 S_n^{(\rho)}/dt^2$.

The proof of (6.7) and (6.8) is likewise immediate if $\rho = 1$, for

$$\begin{aligned} S_n^{(1)} &= \frac{1}{2(n+1)} \left\{ \frac{\sin \frac{1}{2}(n+1)t}{\sin \frac{1}{2}t} \right\}^2 \\ &= \frac{\sin\{(n+1)t - \frac{1}{2}\pi\}}{A_n^{(1)} (2 \sin \frac{1}{2}t)^2} + \frac{1}{4(n+1)(\sin \frac{1}{2}t)^2}, \end{aligned}$$

and so $R_n^{(1)} = \{4(n+1)(\sin \frac{1}{2}t)^2\}^{-1}$.

Consider, then, the values of ρ on the interval $-1 < \rho < 1$. We let

$$\sigma_n^{(\rho)}(z) = \sum_{\nu=0}^{n} A_\nu^{(\rho)} z^\nu,$$

and observe that $S_n^{(\rho)}$ may be written in the form

$$S_n^{(\rho)}(t) = \frac{1}{A_n^{(\rho)}} \Re[\tfrac{1}{2} e^{int} \{\sigma_n^{(\rho)}(e^{-it}) + \sigma_{n-1}^{(\rho)}(e^{-it})\}], \tag{7.2}$$

where $i = \sqrt{(-1)}$ and $\Re\{U(z)\}$ denotes the real part of $U(z)$. We next set

$$r_n^{(\rho)}(z) = (1-z)^{1-\rho} - (1-z)^2 \sigma_n^{(\rho)}(z),$$

choosing that branch of the function $(1-z)^{1+\rho}$ which is equal to $+1$ at the origin, and find, upon using the identity

$$1 - e^{-it} = 2e^{i(\pi - t)/2} \sin \tfrac{1}{2}t,$$

that (7.2) may be written as

$$\begin{aligned} A_n^{(\rho)} S_n^{(\rho)} &= \Re\left[\frac{e^{int}}{(1-e^{-it})^{i+\rho}}\right] + \frac{1}{8(\sin \frac{1}{2}t)^2} \Re[e^{i(n+1)t} \{r_n^{(\rho)}(e^{-it}) + r_{n-1}^{(\rho)}(e^{-it})\}] \\ &= \frac{\sin\{(n + \frac{1}{2} + \frac{1}{2}\rho)t - \frac{1}{2}\rho\pi\}}{(2 \sin \frac{1}{2}t)^{1+\rho}} + \frac{1}{8(\sin \frac{1}{2}t)^2} \Re[e^{i(n+1)t} \{r_n^{(\rho)}(e^{-it}) + r_{n-1}^{(\rho)}(e^{-it})\}]. \end{aligned}$$

The first term on the right here is $A_n^{(\rho)}T_n^{(\rho)}$, by definition. Hence

$$R_n^{(\rho)}(t) = \frac{1}{8A_n^{(\rho)}(\sin\frac{1}{2}t)^2}\Re[e^{i(n+1)t}\{r_n^{(\rho)}(e^{-it})+r_{n-1}^{(\rho)}(e^{-it})\}].$$

Now, employing (6.9), we get

$$r_n^{(\rho)}(z)$$
$$=(2A_n^{(\rho)}-A_{n-1}^{(\rho)}+A_{n+1}^{(\rho-2)})z^{n+1}+(A_{n+2}^{(\rho-2)}-A_n^{(\rho)})z^{n+2}+\sum_{\nu=n+3}^{\infty}A_\nu^{(\rho-2)}z^\nu$$

for all $|z|\leqslant 1$, except at the point $z=1$. We thus deduce that for $0<t<2\pi$

$$R_n^{(\rho)}(t)=\frac{V_n^{(\rho)}(t)}{4A_n^{(\rho)}(\sin\frac{1}{2}t)^2},$$

where $V_n^{(\rho)}(t)=A_{n+1}^{(\rho-1)}+\sum_{\nu=1}^{\infty}A_{n+\nu+1}^{(\rho-2)}\cos\nu t.$

This equation gives (6.7) immediately if we use (7.1). For we have

$$|R_n^{(\rho)}|<\frac{M}{n^\rho t^2}\left\{(n+1)^{\rho-1}+\sum_{\nu=1}^{\infty}(n+\nu+1)^{\rho-2}\right\}<\frac{M}{nt^2}.$$

It also gives (6.8) if we differentiate after summing in Abel's manner. In fact,

$$V_n^{(\rho)}=A_{n+1}^{(\rho-1)}-\tfrac{1}{2}A_{n+2}^{(\rho-2)}-\frac{1}{2\sin\frac{1}{2}t}\sum_{\nu=1}^{\infty}A_{n+\nu+2}^{(\rho-3)}\sin(\nu+\tfrac{1}{2}t),$$

and so

$$\left|\frac{dV_n^{(\rho)}}{dt}\right|<\frac{M}{t^2}\sum_{\nu=1}^{\infty}(n+\nu+2)^{\rho-3}+\frac{M}{t}\sum_{\nu=1}^{\infty}\nu(n+\nu+2)^{\rho-3}.$$

Hence
$$\left|\frac{dR_n^{(\rho)}}{dt}\right|<\frac{M|V_n^{(\rho)}|}{n^\rho t^3}+\frac{M}{n^\rho t^2}\left(\frac{1}{n^{2-\rho}t^2}+\frac{1}{n^{1-\rho}t}\right)$$
$$<Mn^{-1}t^{-3}+Mn^{-2}t^{-4}.$$

This completes the proof.

8. Further lemmas for Theorem II. 8.1. In the rest of the paper y has the value
$$y=\pi/(n+\tfrac{1}{2}\rho+\tfrac{1}{2}),$$
and N is a constant which may depend upon m and ρ and other parameters, but is independent of n and k.

LEMMA 7. *If $kx\leqslant v$, then*

$$\int_{kx}^{v}|\Delta_x^{(r+m)}\phi_r(t)|\,dt\leqslant x^r(v+rx)^{1+\rho}\int_{kx}^{v+rx}\frac{|\Delta_x^{(m)}\phi(t)|}{t^{1+\rho}}\,dt$$

for every pair of integers $r\geqslant 0$ and $m\geqslant 1$.

The proof is in principle the same as that of Lemma 2. In the first place, the relation plainly holds when r is zero and m is arbitrary. In the second place, assuming it holds when r is replaced by $r-1$ and m is arbitrary, we have

$$\begin{aligned}\int_{kx}^{v} |\Delta_x^{(r+m)}\phi_r(t)|\,dt &= \int_{kx}^{v} \left| \int_{t}^{t+x} \Delta_x^{(r+m-1)}\phi_{r-1}(u)\,du \right| dt \\ &\leqslant x \int_{kx}^{v+x} |\Delta_x^{(r+m-1)}\phi_{r-1}(u)|\,du \\ &\leqslant x^r(v+rx)^{1+\rho} \int_{kx}^{v+rx} \frac{|\Delta_x^{(m)}\phi(t)|}{t^{1+\rho}}\,dt.\end{aligned}$$

The lemma then follows by induction.

8.2. LEMMA 8. *If* (6.1) *holds for some* $-1<\rho\leqslant 0$, *and* $\phi_r(x)=o(x^r)$ *for some integer* $r\geqslant 0$, *then* $\phi_1(x)=o(x)$. *Further, if* (6.1) *holds for some* $\rho>0$ *and* $\phi_r(x)=O(x^r)$, *then* $\phi_1(x)=O(x)$.

We prove that (6.1), with $-1<\rho\leqslant 0$, and

$$\phi_{r+1}(x)=o(x^{r+1}) \tag{8.21}$$

imply

$$\phi_r(x)=o(x^r), \tag{8.22}$$

and that (6.1), with $\rho>0$, and

$$\phi_{r+1}(x)=O(x^{r+1}) \tag{8.23}$$

imply

$$\phi_r(x)=O(x^r). \tag{8.24}$$

We first observe that

$$hh_1x\phi_r(x)=G(x,k)+\int_{h_1x}^{hx} dt \int_{kt}^{x-(m+r-1)t} \Delta_t^{(m+r-1)}\phi_{r-1}(u)\,du, \tag{8.25}$$

where $h=(k+r+m-1)^{-1}$, $h_1=(k+r+m)^{-1}$, and $G(x,k)$ is of the form

$$G(x,k)=\sum_{\nu=0}^{r+m-1}\psi\phi_{r+1}\{h(k+\nu)x\}+\sum_{\nu=0}^{r+m-1}\psi\phi_{r+1}\{h_1(k+\nu)x\},$$

the ψ's being independent of x. This relation can, in fact, be verified in the same way that (2.42) was.

We next observe that, because of (8.25) and Lemma 7,

$$\frac{|\phi_r(x)|}{x^r}\leqslant\frac{|G(x,k)|}{hh_1x^{r+1}}+\frac{x^{\rho-r}}{hh_1}\int_{h_1x}^{hx} t^{r-1-\rho}\eta_\rho^{(m)}(t,k)\,dt \tag{8.26}$$

for $0<x\leqslant\pi$.

Suppose, then, that (6.1), with $-1<\rho\leqslant 0$, and (8.21) hold. Under these circumstances, the first term on the right of (8.26) is $o(1)$ for every k, while the second does not exceed the least upper bound of $\eta_\rho^{(m)}(x', k)$ for x' on the interval $0<x'\leqslant hx$, and is therefore $\bar{o}(1)$. Hence (8.22) holds.

Suppose, on the other hand, that (6.1), with $\rho>0$, and (8.23) hold. Then for some k, we have $\eta_\rho^{(m)}(x, k)=O(1)$, and hence also

$$\frac{|\phi_r(x)|}{x^r}\leqslant O(1)+O\Big(x^{\rho-r}\int\limits_{h_1x}^{hx} t^{r-1-\rho}\,dt\Big)=O(1).$$

This is (8.24), and completes the proof.

8.3. Lemma 9. *If $\phi_r(x)=O(x^r)$ and $\phi_{r_1}(x)=o(x^{r_1})$ for some pair of numbers $r_1\geqslant r\geqslant 0$, then $\phi_{r+\delta}(x)=o(x^{r+\delta})$ for every positive δ.*

This is a particular case of a theorem of M. Riesz.†

8.4. Lemma 10. *If $r\geqslant 0$ and $\phi_r(x)=o(x^r)$, the series* (1.1) *is summable* (C, $r+\delta$), *to sum s, for every positive δ.*

This theorem for the case $r=0$ is due to Riesz,‡ for the case $0<r<1$ to Hardy and Littlewood,§ for the case $r=1$ to Young,‖ and for the case $r>1$ to Bosanquet.†† All we need here is Bosanquet's result.

8.5. Lemma 11. *If $1<\rho\leqslant 0$ and* (C_1) *holds, or if $0<\rho\leqslant 1$ and* (E) *holds, then*

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\left|\int\limits_0^{(k+\nu)y}\phi(t)\,S_n^{(\rho)}(t)\,dt\right|=0 \tag{8.51}$$

for every ν.

It is enough to prove the lemma with $\nu=0$, since $k+\nu$ becomes infinite with k. We have, upon integrating by parts,

$$\left|\int\limits_0^{ky}\phi(t)\,S_n^{(\rho)}(t)\,dt\right|\leqslant|\phi_1(ky)S_n^{(\rho)}(ky)|+\left|\int\limits_0^{ky}\phi_1(t)\frac{ds_n^{(\rho)}}{dt}\,dt\right|$$
$$=J(n,k)+K(n,k),$$

say.

† M. Riesz, **16**. See p. 7.

‡ M. Riesz, **16** and **18**. Chapman, **2**, gives another proof of the same fact.

§ Hardy and Littlewood, **9**.

‖ Young, **24**. Young's result is a generalization of Lebesgue's, **13**, that (C_1) implies summability (C, 2).

†† Bosanquet, **1**. Bosanquet's result was stated but not proved by Hardy and Littlewood, **9**.

Now, if $-1<\rho\leqslant 0$ and (C_1) holds,

$$\overline{\lim_{n\to\infty}} J(n,k)\leqslant N\overline{\lim_{n\to\infty}}|n\phi_1(ky)|=0,$$

$$\overline{\lim_{n\to\infty}} K(n,k)\leqslant N\overline{\lim_{n\to\infty}}\left(n^2\int_0^{ky}|\phi_1(t)|\,dt\right)=0,$$

by (6.4) and (6.5). Hence (8.51) is established in this case.

On the other hand, if $0<\rho\leqslant 1$ and (E) holds, then

$$\begin{aligned}J(n,k)&<Nky\{|T_n^{(\rho)}(ky)|+|R_n^{(\rho)}(ky)|\}\\&<Nky\{n^{-\rho}(ky)^{-1-\rho}+n^{-1}(ky)^{-2}\}<Nk^{-\rho}\end{aligned}\tag{8.52}$$

by (6.7), while

$$\begin{aligned}\overline{\lim_{n\to\infty}} K(n,k)&\leqslant\overline{\lim_{n\to\infty}}\left|\left[\phi_2(t)\frac{ds_n^{(\rho)}}{dt}\right]_0^t-\int_0^{ky}\phi_2(t)\frac{d^2s_n^{(\rho)}}{dt^2}\,dt\right|\\&\leqslant N\overline{\lim_{n\to\infty}}\left\{n^2|\phi_2(ky)|+n^3\int_0^{ky}|\phi_2(t)|\,dt\right\}=0,\end{aligned}\tag{8.53}$$

by (6.5) and (6.6). From (8.52) and (8.53) we deduce (8.51).

8.6. Lemma 12. *If* $\phi_1(x)=O(x)$, *then*

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\left|\int_{(k+\nu)y}^{\pi+\mu y}\phi(t)R_n^{(\rho)}(t)\,dt\right|=0$$

for every μ *and* ν.

We may suppose again that $\nu=0$. We then have

$$\left|\int_{ky}^{\pi+my}\phi(t)R_n^{(\rho)}(t)\,dt\right|=\left|[\phi_1(t)R_n^{(\rho)}(t)]_{ky}^{\pi+\mu y}-\int_{ky}^{\pi+\mu y}\phi_1(t)\frac{dR_n^{(\rho)}}{dt}\,dt\right|$$

$$<\frac{N}{n}+\frac{N}{k}+\frac{N}{n}\int_{ky}^{\infty}\frac{dt}{t^2}<\frac{N}{n}+\frac{N}{k}$$

for y small enough. The lemma then clearly follows.

8.7. Lemma 13. *If* $\phi_1(x)=O(x)$, *then*

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\left|\frac{1}{A_n^{(\rho)}}\int_{ky}^{\pi-\frac{1}{2}my}\phi(t+\nu y)\omega\sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right)dt\right|=0\tag{8.71}$$

for every $m \geqslant \nu > 0$, where

$$\omega = \omega(y,t,\rho,\nu,m) = \frac{m}{\{\sin\frac{1}{2}(t+\nu y)\}^{1+\rho}} - \frac{m-\nu}{(\sin\frac{1}{2}t)^{1+\rho}} - \frac{\nu}{\{\sin\frac{1}{2}(t+my)\}^{1+\rho}}.$$

We first prove that

$$|\omega| < \frac{My^2}{t^{3+\rho}}, \qquad \left|\frac{d\omega}{dt}\right| < \frac{My^2}{t^{4+\rho}} \tag{8.72), (8.73}$$

for $my \leqslant \pi$ and $0 < t \leqslant \pi - \frac{1}{2}my$, where the M's are independent of t and y. Consider the function

$$\omega_1(x) = \frac{m}{\{\sin\frac{1}{2}(t+\nu x)\}^{1+\rho}} - \frac{m-\nu}{(\sin\frac{1}{2}t)^{1+\rho}} - \frac{\nu}{\{\sin\frac{1}{2}(t+mx)\}^{1+\rho}}$$

as a function of x, for a fixed t on the interval $0 < t < \pi$. Evidently ω_1 vanishes with its first derivative

$$\omega_1'(x) = -\tfrac{1}{2}m\nu(1+\rho)\left[\frac{\cos\frac{1}{2}(t+\nu x)}{\{\sin\frac{1}{2}(t+\nu x)\}^{2+\rho}} - \frac{\cos\frac{1}{2}(t+mx)}{\{\sin\frac{1}{2}(t+mx)\}^{3+\rho}}\right]$$

at $x = 0$. Moreover,

$$|\omega_1''(x)| = \tfrac{1}{4}m\nu(1+\rho)\left|\frac{\nu}{\{\sin\frac{1}{2}(t+\nu x)\}^{1+\rho}} + \frac{\nu(2+\rho)\{\cos\frac{1}{2}(t+\nu x)\}^2}{\{\sin\frac{1}{2}(t+\nu x)^{3+\rho}} - \right.$$

$$\left. - \frac{m}{\{\sin\frac{1}{2}(t+mx)\}^{1+\rho}} - \frac{m(2+\rho)\{\cos\frac{1}{2}(t+mx)\}^2}{\{\sin\frac{1}{2}(t+mx)\}^{2+\rho}}\right|$$

$$\leqslant 2m^3[\{\operatorname{cosec}(t+\nu x)\}^{3+\rho} + \{\operatorname{cosec}\tfrac{1}{2}(t+mx)\}^{3+\rho}] \leqslant 4m^3(3\pi/t)^{3+\rho}$$

for $0 < t \leqslant \pi - \frac{1}{2}mx$ and $mx \leqslant \pi$. Accordingly,

$$|\omega(y,t,\rho,\nu,m)| = |\omega_1(y)| = \left|\int_0^y du \int_0^u \omega''(v)\,dv\right| < \frac{Ny^2}{t^{3+\rho}}.$$

This is (8.72). Similarly, the function

$$\omega_2(x) = -\tfrac{1}{2}(1+\rho)\left[\frac{m\cos\frac{1}{2}(t+\nu x)}{\{\sin\frac{1}{2}(t+\nu x)\}^{2+\rho}} - \frac{(m-\nu)\cos\frac{1}{2}t}{(\sin\frac{1}{2}t)^{2+\rho}} - \frac{\nu\cos\frac{1}{2}(t+mx)}{\{\sin\frac{1}{2}(t+mx)\}^{2+\rho}}\right],$$

which for a fixed $x = y$ is the derivative of ω with regard to t, vanishes with its derivative at $x = 0$. Moreover, it is easily verified that

$$|\omega_2''(x)| \leqslant 3m^3(3\pi/t)^{4+\rho}$$

for $0 < t \leqslant \pi - \frac{1}{2}mx$ and $mx \leqslant \pi$. This plainly gives (8.73).

The proof of the lemma is now immediate. Integrating by parts, we obtain

$$\left|\frac{1}{A_n^{(\rho)}}\int_{ky}^{\pi-\frac{1}{2}my}\phi(t+\nu y)\omega\sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right)dt\right|$$

$$=\frac{1}{A_n^{(\rho)}}\left|\left[\phi_1(t+\nu y)\omega\sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right)\right]_{ky}^{\pi-\frac{1}{2}my}-\right.$$

$$-\int_{ky}^{\pi-\frac{1}{2}my}\phi_1(t+\nu y)\frac{d\omega}{dt}\sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right)dt-$$

$$\left.-\frac{\pi}{y}\int_{ky}^{\pi-\frac{1}{2}my}\phi_1(t+\nu y)\omega\cos\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right)dt\right|$$

$$<Ny^{2+\rho}+\frac{N}{k^{2+\rho}}+Ny^{1+\rho}\int_{ky}^{\pi}\frac{dt}{t^{2+\rho}}<Ny^{2+\rho}+\frac{N}{k^{1+\rho}}$$

for y small enough. This last function is annihilated by the operator $\lim_{k\to\infty}\overline{\lim_{n\to\infty}}$; and the lemma is established.

8.8. LEMMA 14. *If m' is an integer $\geqslant m$, and $\eta_\rho^{(m)}(x,k)=\bar{o}(1)$, then* $\eta_\rho^{(m')}(x,k)=\bar{o}(1)$.

It is sufficient to prove the lemma with $m'=m+1$. We have, since $f(t)$ is even and periodic,

$$\eta_\rho^{(m+1)}(x,k)\leqslant x^\rho\int_{kx}^{\pi}\frac{|\Delta_x^{(m)}f(t+x)|}{t^{1+\rho}}dt+\eta_\rho^{(m)}(x,k)$$

$$<M\eta_\rho^{(m)}(x,k)+x^\rho\int_{\pi}^{\pi+x}\frac{|\Delta_x^{(m)}f(t)|}{(t-x)^{1+\rho}}dt$$

$$<M\eta_\rho^{(m)}(x,k)+x^\rho\int_{\pi-x}^{\pi}\frac{|\Delta_x^{(m)}f(t)|}{(2\pi-t-x)^{1+\rho}}dt<M\eta_\rho^{(m)}(x,k)$$

for x small enough, where the M's are independent of x and k. The lemma evidently follows.

9. Proof of Theorem II. If (C) holds, and if (6.1) is satisfied with ρ some number in the interval $-1<\rho\leqslant 0$, then (C_1) holds, by

Lemmas 1 and 8. Moreover, if (C) holds, and (6.1) is satisfied with some $\rho > 0$, then (E) holds, by Lemmas 1, 8, and 9. Accordingly, the proof reduces to showing that (C_1) and (6.1), with $-1 < \rho \leqslant 0$, or (E) and (6.1), with $\rho > 0$, imply summability (C, ρ), to sum s.

The case $\rho > 1$ offers no difficulty. If (E) holds, then

$$\phi^{(x)}_{\frac{1}{2}(1+\rho)} = o(x^{\frac{1}{2}(1+\rho)}), \tag{9.1}$$

by Lemma 9; and (9.1) implies summability (C, ρ), by Lemma 10.

Suppose, then, that $-1 < \rho \leqslant 1$. The nth Cesàro mean $s_n^{(\rho)}$ of order ρ of the series (1.1) is given by the formula

$$s_n^{(\rho)} - s = \frac{2}{\pi}\int\limits_0^{\pi} \phi(t) S_n^{(\rho)}(t)\,dt.$$

This we may write

$$\begin{aligned}
&2^{2m-1}\pi(s_n^{(\rho)}-s) \\
&\quad = \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int\limits_0^{(k+\nu)y}\phi(t)S_n^{(\rho)}(t)\,dt + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int\limits_{(k+\nu)y}^{\pi+(\nu-m)y}\phi(t)R_n^{(\rho)}(t)\,dt + \\
&\qquad + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int\limits_{\pi+(\nu-m)y}^{\pi}\phi(t)S_n^{(\rho)}(t)\,dt + \sum_{\nu=0}^{2m}\binom{2m}{\nu}\int\limits_{(k+\nu)y}^{\pi+(\nu-m)y}\phi(t)T_n^{(\rho)}(t)\,dt \\
&\quad = \alpha(n,k)+\beta(n,k)+\gamma(n,k)+\delta(n,k),
\end{aligned}$$

say.

Now (C_1) or (E) implies both

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\,|\alpha(n,k)| = 0, \tag{9.2}$$

and

$$\lim_{k\to\infty}\overline{\lim_{n\to\infty}}\,|\beta(n,k)| = 0, \tag{9.3}$$

by Lemmas 11 and 12.

Furthermore, since $\phi(t)S_n^{(\rho)}(t)$ is even and periodic, we have

$$\begin{aligned}
&\gamma(n,k) \\
&\quad = \sum_{\nu=0}^{m-1}\binom{2m}{\nu}\int\limits_{\pi+(\nu-m)y}^{\pi}\phi(t)S_n^{(\rho)}(t)\,dt + \sum_{\nu=m+1}^{2m}\binom{2m}{\nu}\int\limits_{\pi+(\nu-m)y}^{\pi}\phi(t)S_n^{(\rho)}(t)\,dt \\
&\quad = \sum_{\nu=0}^{m-1}\binom{2m}{\nu}\int\limits_{\pi}^{\pi+(m-\nu)y}\phi(t)S_n^{(\rho)}(t)\,dt - \sum_{\nu=m+1}^{2m}\binom{2m}{\nu}\int\limits_{\pi}^{\pi+(\nu-m)y}\phi(t)S_n^{(\rho)}(t)\,dt \\
&\quad = 0.
\end{aligned} \tag{9.4}$$

Hence it remains to consider $\delta(n,k)$. We write

$$\delta(n,k) = \sum_{\nu=0}^{2m} \binom{2m}{\nu} \int_{ky}^{\pi-my} \phi(t+\nu y) T_n^{(\rho)}(t+\nu y)\,dt$$

$$= \frac{1}{2^{1+\rho}A_n^{(\rho)}} \left[\int_{ky}^{\pi-my} \frac{\Delta y^{(2m-1)}\phi(t+y)}{\{\sin\frac{1}{2}(t+2my)\}^{1+\rho}} \sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right) dt - \right.$$

$$- \int_{ky}^{\pi-my} \frac{\Delta y^{(2m-1)}\phi(t)}{(\sin\frac{1}{2}t)^{1+\rho}} \sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right) dt +$$

$$\left. + \sum_{\nu=1}^{2m-1} \frac{(-1)^\nu}{2m} \binom{2m}{\nu} \int_{ky}^{\pi-my} \phi(t+\nu y)\omega \sin\left(\frac{t\pi}{y}-\frac{\rho\pi}{2}\right) dt \right],$$

setting

$$\omega = \omega(y,t,\rho,\nu,2m)$$

$$= \frac{2m}{\{\sin\frac{1}{2}(t+\nu y)\}^{1+\rho}} - \frac{2m-\nu}{(\sin\frac{1}{2}t)^{1+\rho}} - \frac{\nu}{\{\sin\frac{1}{2}(t+2my)\}^{1+\rho}}.$$

It is then seen from Lemma 13 that

$$\overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}}\ |\delta(n,k)| \leqslant N\ \overline{\lim_{k\to\infty}}\ \overline{\lim_{n\to\infty}}\ \eta_\rho^{(2m-1)}(y,k),$$

if either (C_1) or (E) holds; and further, from Lemma 14, that

$$\lim_{k\to\infty}\ \overline{\lim_{n\to\infty}}\ |\delta(n,k)| = 0 \tag{9.5}$$

if (6.1) likewise holds.

The theorem now follows from (9.2), (9.3), (9.4), and (9.5).

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A NEW WARING'S PROBLEM WITH SQUARES OF LINEAR FORMS

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1. LET
$$f(X, Y) = aX^2 + 2hXY + bY^2 \tag{1}$$
where $a \geqslant 0$, h, b are given integers and
$$\Delta = ab - h^2 \geqslant 0, \tag{1.1}$$
so that the quadratic form $f(X, Y)$ is never negative for real values of X, Y. The new problem, which apparently has neither been mentioned nor considered by other writers, is to find the least value of n independent of a, h, b, say N, such that integers
$$a_r, b_r \qquad (r = 1, 2, \ldots, n)$$
exist for which
$$f(X, Y) = \sum_{r=1}^{n} (a_r X + b_r Y)^2 \tag{1.2}$$
identically in X, Y.

For one variable when $b = h = 0$ and also when $\Delta = 0$, $N = 4$ as the result is simply Bachet's theorem, first proved by Lagrange, that every positive integer is the sum of the squares of four integers.

It is obvious that $N \geqslant 5$, on considering the case
$$X^2 + 7Y^2 = X^2 + Y^2 + Y^2 + Y^2 + 4Y^2,$$
as then $a_1 = 1$, $a_2 = a_3 = \ldots = 0$, $b_1 = 0$ and
$$7 = b_2^2 + b_3^2 + b_4^2 + b_5^2 + \ldots$$
is not possible with less than four squares. In a paper to appear in the *Mathematische Zeitschrift*,* I proved that $N \leqslant 6$ if $h = 0$, $N \leqslant 8$ if $h \neq 0$. The proof depended on the classic elementary theory of numbers, and was obviously not a method for finding the value of N. The difficulty was to find any non-trivial upper bound for N, the value $N \leqslant 12$ being obvious on taking $f(X, Y)$ to be a reduced form, i.e. $a \geqslant b \geqslant 2|h|$. For
$$f(X, Y) = (a - |h|)X^2 + |h|(X \pm Y)^2 + (b - |h|)Y^2,$$
and four squares suffice for each of these three terms.

I also investigated there a different problem, namely, for *given* $n = 2, 3, 4, 5 \ldots$ in (1.2), to find the necessary and sufficient con-

* 'On the representation of a binary quadratic form as a sum of squares of linear forms.'

ditions for the existence of *rational* values of a_r, b_r. It was shown that when $n = 2$, integral values of a_r, b_r existed if rational ones did; and similarly when $n = 3$ provided $[a,h,b] = 1$, i.e. the greatest common divisor of a, h, b is one. The condition for rational solutions when $n = 4$ was shown to be that

$$\Delta \neq 4^\rho(8\sigma+7) \tag{1.3}$$

where $\rho \geqslant 0$, $\sigma \geqslant 0$ are integers, i.e. that Δ can be expressed as a sum of three integer squares. It had already been shown by Landau* that rational solutions always exist when $n = 5$ and so of course if $n > 5$.

This suggests that integer values of a_r, b_r exist when $n = 4$, if and only if (1.3) holds, and that they are always possible if $n = 5$. I have now found a proof depending curiously enough upon a type of theorem first proved by me† some thirteen or fourteen years ago and dealing with the derivation of the representations of a number as a sum of four squares from those of the factors of the number.

The proof of the subsidiary theorem is given in § 2; the case $n = 4$ in § 3, and $n = 5$ in § 4. Section 5 is an application of § 3 concerned with the conditions under which numbers can be represented by certain ternary quadratic forms, while § 6 is a generalization of § 4.

2. It is well known that if l is a positive integer, Jacobi has proved that the number of solutions of

$$l = x_1^2+y_1^2+z_1^2+t_1^2 \tag{2}$$

in integers x_1, y_1, z_1, t_1 is

$$\lambda_l \sum d_l,$$

where the sum refers to all the odd divisors d_l of l, and $\lambda_l = 8$ or 24 according as l is odd or even.

If

$$m = x_2^2+y_2^2+z_2^2+t_2^2, \tag{2.1}$$

where m, x_2, y_2, z_2, t_2 are integers, then integer solutions of

$$lm = x^2+y^2+z^2+t^2 \tag{2.2}$$

are given by Euler's formula

$$\left.\begin{aligned} x &= x_1x_2-y_1y_2-z_1z_2-t_1t_2 \\ y &= x_1y_2+y_1x_2+z_1t_2-t_1z_2 \\ z &= x_1z_2-y_1t_2+z_1x_2+t_1y_2 \\ t &= x_1t_2+y_1z_2-z_1y_2+t_1x_2. \end{aligned}\right\} \tag{2.3}$$

* 'Landau, 'Über die Zerlegung definiter Funktionen in Quadrate': *Archiv der Math. und Physik*, (3), 7 (1904), 21.

† 'On the solutions of $x^2+y^2+z^2+t^2 = 4m_1m_2$': *Messenger of Math.*, 67 (1918), 142–4.

The subsidiary theorem is that if $[l, m] = 1$ or 2, *all* the solutions of (2.2) are given by (2.3), each solution occurring 8 or 24 times according as $[l, m] = 1$ or 2. The method of proof is the same as for my old result. Thus a unique correspondence can be established between a solution x_1, y_1, z_1, t_1 of (2) and the linear substitution of determinant l in two variables.

$$S_1 = \begin{pmatrix} x_1+iy_1, & z_1+it_1 \\ -z_1+it_1, & x_1-iy_1 \end{pmatrix},$$

where the substitution $\begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}$

replaces X, Y by $\alpha X+\beta Y$, $\gamma X+\delta Y$ respectively.

Similarly for $S_2 = \begin{pmatrix} x_2+iy_2, & z_2+it_2 \\ -z_2+it_2, & x_2-iy_2 \end{pmatrix}$,

whence $S_1 S_2 = \begin{pmatrix} x+iy, & z+it \\ -z+it, & x-iy \end{pmatrix} = S$,

since from (2.3)

$$x+iy = (x_1+iy_1)(x_2+iy_2)+(z_1+it_1)(-z_2+it_2)$$
$$z+it = (x_1+iy_1)(z_2+it_2)+(z_1+it_1)(x_2-iy_2).$$

Suppose now $x_1', y_1',\ldots$ and $x_2', y_2',\ldots$ are other solutions of (2), (2.1) giving new substitutions S_1', S_2' and new values of $x, y,\ldots$ in (2.3), say $x', y',\ldots$. We wish to investigate when

$$x' = x, \qquad y' = y, \qquad z' = z, \qquad t' = t,$$

i.e. $\qquad S_1 S_2 = S_1' S_2' \quad$ or $\quad S_1'^{-1} S_1 = S_2' S_2^{-1}$,

where S_2^{-1} is the substitution inverse to S_2, etc. Now $S'^{-1}S_1$ is a substitution of determinant unity whose coefficients are rational numbers with a possible denominator l. So the denominators in $S_2' S_2^{-1}$ are possibly m. Since $[l, m] = 1$ or 2, the denominator can only be 1 or 2.

Suppose first that $[l, m] = 1$. Then

$$S_1'^{-1} S_1 = \begin{pmatrix} \xi+i\eta, & \zeta+i\tau \\ -\zeta+i\tau, & \xi-i\eta \end{pmatrix},$$

where ξ, η, ζ, τ are integers and

$$\xi^2+\eta^2+\zeta^2+\tau^2 = 1.$$

Hence $\xi = \pm 1$, $\eta = 0$, $\zeta = 0$, $\tau = 0$, etc., giving eight solutions, i.e. a value of x, y, z, t occurring once in (2.3) occurs exactly eight times. But there are $\lambda_l \sum d_l$, $\lambda_m \sum d_m$ sets of values of $x_1, y_1,\ldots$ and $x_2, y_2,\ldots$, and so there are

$$\tfrac{1}{8}\lambda_l\lambda_m \sum d_l \sum d_m = \lambda_{lm} \sum d_{lm}$$

different sets of values of x, y, z, t. For, since l and m have no odd factors in common,

$$\sum d_l \sum d_m = \sum d_{lm}.$$

Also either

$$\lambda_l = 8, \qquad \lambda_m = 8, \qquad \lambda_{lm} = 8,$$

or

$$\lambda_l = 8, \qquad \lambda_m = 24, \qquad \lambda_{lm} = 24,$$

or

$$\lambda_l = 24, \qquad \lambda_m = 8, \qquad \lambda_{lm} = 24,$$

and so

$$\lambda_l\lambda_m = 8\lambda_{lm}.$$

Suppose next $[l, m] = 2$. Then

$$S_1'^{-1}S_1 = \begin{pmatrix} \tfrac{1}{2}\xi+\tfrac{1}{2}i\eta, & \tfrac{1}{2}\zeta+\tfrac{1}{2}i\tau \\ -\tfrac{1}{2}\zeta+\tfrac{1}{2}i\tau, & \tfrac{1}{2}\xi-\tfrac{1}{2}i\eta \end{pmatrix}$$

where ξ, η, ζ, τ are integers and

$$\xi^2+\eta^2+\zeta^2+\tau^2 = 4.$$

Hence

$$\xi = \pm 2, \qquad \eta = 0, \qquad \zeta = 0, \qquad \tau = 0, \text{ etc.},$$

giving eight solutions, and

$$\xi^2 = 1, \qquad \eta^2 = 1, \qquad \zeta^2 = 1, \qquad \tau^2 = 1,$$

giving sixteen more solutions, e.g.

$$\xi = 1, \qquad \eta = 1, \qquad \zeta = 1, \qquad \tau = 1.$$

These really give integer solutions for x_1', y_1',..., for then

$$x_1'+iy_1' = \tfrac{1}{2}(x_1+iy_1)(1+i)+\tfrac{1}{2}(z_1+it_1)(1+i),$$

etc., where from (2), since l is even,

$$x_1+y_1+z_1+t_1 \equiv 0 \pmod 2.$$

Hence a value of x, y, z, t occurring once in (2.3) occurs twenty-four times. The same result as before follows since here

$$\lambda_l = \lambda_m = \lambda_{lm} = 24, \qquad \tfrac{1}{24}\lambda_l\lambda_m = \lambda_{lm}.$$

3. It will now be proved that integer values of a_r, b_r, $(r = 1, 2, 3, 4)$ exist such that

$$aX^2+2hXY+bY^2 = \sum_{r=1}^{4} (a_rX+b_rY)^2 \tag{3}$$

identically in X, Y, i.e.

$$a = \sum_{r=1}^{4} a_r^2, \qquad b = \sum_{r=1}^{4} b_r^2, \qquad h = \sum_{r=1}^{4} a_rb_r, \tag{3.1}$$

if and only if $a \geqslant 0$, $\quad 0 \leqslant \Delta = ab-h^2 \neq 4^\rho(8\sigma+7)$, (3.2)

where $\rho \geqslant 0$, $\sigma \geqslant 0$ are integers.

The condition (3.2) is necessary. For we have identically, as a variant of (2.3),

$$\sum_{r=1}^{4} a_r^2 \sum_{r=1}^{4} b_r^2 = \sum_{r=1}^{4} c_r^2,$$

where
$$c_1 = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 = h,$$
$$c_2 = a_1b_2 - a_2b_1 - a_3b_4 + a_4b_3, \text{ etc.}$$

and so Δ must be a sum of three squares.

The proof of the sufficiency of (3.2) is simplified by some preliminary considerations. If (3) is proved for any quadratic form (1), it clearly holds for any form derived from (1) by a linear substitution with integer coefficients and any determinant. We may also suppose that the form (1) is properly primitive, i.e. $[a, h, b] = 1$. For if $[a, h, b] = d > 0$, then d is a sum of squares of four integers. Hence, if (3) holds for the form

$$\frac{a}{d}X^2 + 2\frac{h}{d}XY + \frac{b}{d}Y^2,$$

Euler's identity shows that (3) holds for the form (1). This does not affect (3.2).

Next we may suppose $[a, b] = 1$ or 2 by effecting, if need be, a linear transformation of determinant unity.

The condition (3.2) states that Δ can be expressed as a sum of three squares or, say,

$$ab = h^2 + U^2 + V^2 + W^2.$$

Hence, from § 2, there must exist integer values of x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2 for which

$$a = x_1^2 + y_1^2 + z_1^2 + t_1^2, \qquad b = x_2^2 + y_2^2 + z_2^2 + t_2^2,$$

while h is one of x, y, z, t in (2.3). Take, for example

$$h = x_1y_2 + y_1x_2 + z_1t_2 - t_1z_2,$$

then

$$f(X, Y) = (x_1X + y_2Y)^2 + (y_1X + x_2Y)^2 + (z_1X + t_2Y)^2 + (t_1X - z_2Y)^2.$$

Hence the proof of the theorem, which is clearly independent of which particular value in (2.3) we take for h.

It is well to emphasize that it has been proved that (3.1) is satisfied by integer values of a_r, b_r if and only if $\Delta \neq 4^\rho(8\sigma+7)$, i.e. the condition $[a, b] = 1$ or 2 used in the proof, is not required.

4. It is now not difficult to prove the theorem that every definite quadratic form (1) can be expressed as a sum of five squares of linear forms with integer coefficients. It suffices to assume that

$$\Delta = 4^\rho(8\sigma+7), \tag{4}$$

as otherwise four squares would suffice. The theorem is true for $\Delta = 7, 15$. For

$$X^2+7Y^2 = X^2+Y^2+Y^2+Y^2+4Y^2,$$
$$2X^2+2XY+4Y^2 = X^2+(X+Y)^2+Y^2+Y^2+Y^2,$$
$$X^2+15Y^2 = X^2+Y^2+Y^2+4Y^2+9Y^2,$$
$$3X^2+5Y^2 = X^2+X^2+X^2+Y^2+4Y^2,$$
$$2X^2+2XY+8Y^2 = (X+2Y)^2+(X-Y)^2+Y^2+Y^2+Y^2,$$
$$4X^2+2XY+4Y^2 = X^2+(X+Y)^2+(X+Y)^2+(X-Y)^2+Y^2.$$

Also any form with Δ equal to 7 or 15 is equivalent to one of the six left-hand forms. We may assume, then, that $\Delta \geqslant 23$. On effecting a linear substitution of determinant unity and writing $-y$ for y if need be, we may suppose that the form (1) is reduced and that $h \geqslant 0$, so that

$$b \geqslant a \geqslant 2h, \qquad a \leqslant 2\sqrt{\Delta/3}. \tag{4.1}$$

Consider first the case $\rho = 0$ so that $\Delta = 8\sigma+7$. Take one of the five squares to be q^2Y^2, where q is an unknown integer such that $|q| \leqslant b^{\frac{1}{2}}$. Then

$$aX^2+2hXY+(b-q^2)Y^2$$

must be a sum of four squares. Hence from § 3

$$0 \leqslant \Delta_1 = \Delta - aq^2 \neq 4^{\lambda}(8\mu+7),$$

say, where $\lambda \geqslant 0$, $\mu \geqslant 0$ are integers. If

$$a \equiv 1, 2, 4, 5, 6 \pmod 8,$$

take $q = 1$ and then

$$\Delta_1 \equiv 6, 5, 3, 2, 1 \pmod 8.$$

This requires $\Delta - a \geqslant 0$ which is satisfied as $a \leqslant 2\sqrt{\Delta/3}$. If

$$a \equiv 3, 7 \pmod 8,$$

take $q = 2$ and then

$$\Delta_1 \equiv 3, 3 \pmod 8.$$

This requires $\Delta \geqslant 4a$, and it suffices if $8\sqrt{\Delta/3} \leqslant \Delta$ which is true as $\Delta \geqslant 23$.

We must still discuss $a \equiv 0 \pmod 8$ so that $a \geqslant 8$.

From

$$ab = \Delta+h^2,$$
$$b \leqslant \tfrac{1}{8}\Delta+h^2/a,$$
$$\leqslant \tfrac{1}{8}\Delta+a/4,$$
$$\leqslant \tfrac{1}{8}\Delta+\tfrac{1}{2}\sqrt{\Delta/3} \leqslant \tfrac{1}{4}\Delta,$$

since $\Delta \geqslant 6$. Hence, unless $b \equiv 0 \pmod 8$, we can take p^2x^2 for one of the five squares where $p = 1$ or 2.

We must next discuss the case

$$\Delta \equiv 7 \pmod 8, \qquad a \equiv 0 \pmod 8, \qquad b \equiv 0 \pmod 8,$$

so that h is odd. We take one of the five squares to be $r^2(X+Y)^2$. Hence

$$(a-r^2)X^2+2(h-r^2)XY+(b-r^2)Y^2$$

is the sum of four squares, and so $r^2 \leqslant a$ and

$$0 \leqslant \Delta_2 = (a-r^2)(b-r^2)-(h-r^2)^2.$$
$$= \Delta - r^2(a+b-2h) \neq 4^\lambda(8\mu+7).$$

Now $r^2 \leqslant a$, $0 \leqslant \Delta_2$ if $r^2 \leqslant h$, for then

$$\Delta_2 \geqslant ab-h^2-h(a+b-2h)$$
$$\geqslant (a-h)(b-h) \geqslant 0$$

from (4.1).

Also

$$\Delta_2 \equiv 7+2hr^2 \pmod 8$$
$$\equiv 7 \pm 2r^2 \pmod 8.$$

We take $r=1$. This requires $1 \leqslant h$, which is true since h is odd. Hence the theorem is proved for $\Delta = 8\sigma+7$.

Suppose next $\Delta = 4^\rho(8\sigma+7)$ $(\rho > 0)$. By a theorem of Lipschitz,* every properly primitive quadratic form of determinant Dp^2, where p is a prime, (and hence of determinant $Dp^{2\alpha}$) can be derived from a properly primitive form of determinant D by a substitution with integer coefficients and determinant p (or p^α in the second case). Hence it suffices to prove our theorem for the improperly primitive forms of determinant Δ, i.e. those with $[a, 2h, b] = 2$. But then h is even since $\Delta = ab-h^2$, and we can write

$$aX^2+2hXY+bY^2 = 2[\tfrac{1}{2}aX^2+2(\tfrac{1}{2}h)XY+\tfrac{1}{2}bY^2].$$

The determinant of the form in brackets is $\frac{1}{4}\Delta$. Hence, step by step, we are brought to the case $\rho = 0$. Hence the theorem is proved and $N = 5$ in § 1.

A similar problem arises for the general positive definite quadratic form in s variables. In the *Zeitschrift* paper, it has been proved that the form can be expressed as a sum of squares of $s+3$ linear forms with rational coefficients. The coefficients are integers when $s=1$ or 2, and it would be interesting to know if this is true in general.

5. We consider now the application of § 3 to the representation of numbers by definite ternary quadratic forms. In the following

* Matthews, *Theory of Numbers*, pp. 159–62.

scheme, the right-hand expressions, wherein $\rho \geqslant 0$, $\sigma \geqslant 0$ are integers, denote those numbers and only those numbers which cannot be represented by the corresponding left-hand quadratic forms for integer values of x, y, z.

$$x^2+y^2+z^2, \qquad 4^\rho(8\sigma+7) \tag{5}$$
$$x^2+y^2+2z^2, \qquad 2^{2\rho+1}(8\sigma+7) \tag{5.1}$$
$$x^2+y^2+3z^2, \qquad 3^{2\rho+1}(3\rho+2) \tag{5.2}$$
$$x^2+2y^2+2z^2, \qquad 4^\rho(8\sigma+7) \tag{5.3}$$
$$x^2+2y^2+3z^2, \qquad 2^{2\rho+1}(8\sigma+5) \tag{5.4}$$
$$x^2+2y^2+4z^2, \qquad 2^{2\rho+1}(8\sigma+7) \tag{5.5}$$
$$x^2+2y^2+5z^2, \qquad 5^{2\rho+1}(5\sigma\pm 2) \tag{5.6}$$
$$x^2+y^2+4z^2, \qquad 4^\rho(8\sigma+7),\ 8\sigma+3 \tag{5.7}$$
$$x^2+y^2+5z^2, \qquad 4^\rho(8\sigma+3) \tag{5.8}$$
$$x^2+y^2+6z^2, \qquad 3^{2\rho+1}(3\sigma+1) \tag{5.9}$$
$$x^2+y^2+8z^2, \qquad 2^{2\rho+1}(8\sigma+7),\ 16\sigma+6,\ 4\sigma+3, \tag{5.10}$$
$$x^2+2y^2+6z^2, \qquad 4^\rho(8\sigma+5) \tag{5.11}$$
$$x^2+2y^2+8z^2, \qquad 4^\rho(8\sigma+7),\ 8\sigma+5 \tag{5.12}$$
$$x^2+3y^2+3z^2, \qquad 9^\rho(3\sigma+2) \tag{5.13}$$
$$x^2+5y^2+5z^2, \qquad 4^\rho(8\sigma+7),\ 5\sigma\pm 2 \tag{5.14}$$
$$2x^2+3y^2+6z^2, \qquad 4^\rho(8\sigma+7),\ 3\sigma+1 \tag{5.15}$$
$$2x^2+3y^2+3z^2, \qquad 9^\rho(3\sigma+1). \tag{5.16}$$

Results of this kind are particular cases in the general arithmetical theory of ternary quadratic forms due to Gauss, Eisenstein, and Smith. This gives necessary and sufficient conditions for the representation of a given number by a genus of forms, i.e. a certain finite number of classes of forms. If there is only one class in the genus, the method gives at once the required conditions for the representation of numbers by the class.

Some of the results tabulated are due to Gauss, Cauchy, Lebesgue, Dirichlet, and Eisenstein, who very likely knew most of them. Excluding (5.13) noted by Dickson and (5.14), (5.15), (5.16) noted by me, these results have been gathered together by Ramanujan,* who most likely found some of them empirically. They are not all independent. Thus (5.3), (5.7), (5.12) are obvious deductions from (5), while (5.5),

* Ramanujan, 'On certain quaternary forms': *Collected Mathematical Works*, 1927, pp. 171, 172. See also my remarks in the Appendix pp. 341, 342.

(5.10) follow easily from (5.1). A proof of (5.2), (5.4), (5.6), (5.9), (5.13) has been given by Dickson* and is based on the general theory.

The theorem of § 3 states that integer solutions a_r, b_r, $(r = 1, 2, 3, 4)$ of

$$a = \sum_{r=1}^{4} a_r^2, \qquad h = \sum_{r=1}^{4} a_r b_r, \qquad b = \sum_{r=1}^{4} b_r^2,$$

exist if, and only if, $0 \leqslant ab - h^2 \neq 4^\rho(8\sigma+7)$ and of course $a \geqslant 0$.

Take $a = 5$, so that

$$a_1 = \pm 2, \qquad a_2 = \pm 1, \qquad a_3 = a_4 = 0,$$

and other solutions derived by permuting these values. The signs are of no importance as $-b_r$ can be written for b_r. Hence

$$h = 2b_1 + b_2, \qquad b = b_1^2 + b_2^2 + b_3^2 + b_4^2.$$

Take $h = 0$, then clearly

$$b = 5b_1^2 + b_3^2 + b_4^2$$

has integer solutions, if and only if $5b \neq 4^\rho(8\sigma+7)$, i.e. b is not of the form $4^\rho(8\sigma+3)$, and this is (5.8).

So $$b = (2b_1 - h)^2 + b_1^2 + b_3^2 + b_4^2,$$
or $$5b - h^2 = (5b_1 - 2h)^2 + 5b_3^2 + 5b_4^2,$$
has integer solutions, if and only if $5b - h^2 \neq 4^\rho(8\sigma+7)$. Write

$$m = x^2 + 5y^2 + 5z^2.$$

On taking residues (mod 5), this certainly has no integer solutions unless $m \equiv 0, \pm 1 \pmod 5$. We can then write $m = 5b - h^2$ where $h = 0, \pm 1, \pm 2$, and clearly we can take the sign of x so that $x \equiv -2h \pmod 5$. Hence integer solutions for x, y, z exist, if and only if $m \equiv 0, \pm 1 \pmod 5$ and $m \neq 4^\rho(8\sigma+7)$. This is (5.14). This result, which includes (5.8), can also be easily proved on noting

$$m = x^2 + (2y+z)^2 + (y-2z)^2,$$

and using (5).

Take next $a = 6$, so that $a_1 = 2$, $a_2 = 1 = a_3$, $a_4 = 0$,

$$h = 2b_1 + b_2 + b_3, \qquad b = b_1^2 + b_2^2 + b_3^2 + b_4^2,$$

whence $$b = b_1^2 + b_2^2 + (h - 2b_1 - b_2)^2 + b_4^2,$$
from which

$$6b - h^2 = 2(3b_1 - h)^2 + 3(2b_1 + 2b_2 - h)^2 + 6b_4^2.$$

This is possible if and only if $0 \leqslant 6b - h^2 \neq 4^\rho(8\sigma+7)$. Now if

$$m = 2x^2 + 3y^2 + 6z^2$$

* Dickson, 'Integers represented by positive ternary quadratic forms': *Bull. American Math. Soc.* **33** (1927), 63–70. (5.9) is proved in his paper 'Quaternary quadratic forms representing all integers': *American J. of Math.* 69 (1927), 39–56, pp. 42–4.

is solvable, on taking residues (mod 6), we find $m \equiv 0, 2, 3, 5 \pmod 6$, i.e. $m = 6b - h^2$ where $0 \leqslant h < 6$. Then the sign of x can be taken so that $x \equiv -h \pmod 3$, i.e. $x = 3b_1 - h$, and as $y \equiv -h \pmod 2$, we can put $y = 2b_1 + 2b_2 - h$. Hence the equation is possible, if and only if $m \equiv 0, 2, 3, 5 \pmod 6$ and $m \neq 4^\rho(8\sigma+7)$, and this is (5.15). In the particular case when $m \equiv 0 \pmod 6$, the result simplifies on writing $6m$ for m, $3x$ for x, $2y$ for y. We see then that

$$m = 3x^2 + 2y^2 + z^2$$

is solvable, if and only if $6m \neq 4^\rho(8\sigma+7)$, i.e. $m \neq 2^{2\rho+1}(8\sigma+5)$, and this is (5.4).

The case $a = 2$ leads to (5.1), while $a = 3$ leads to a result included in $a = 6$. The results (5.2), (5.6), (5.13) cannot apparently be deduced in this way.

An interesting result follows from $a = 4$. Then

$$a_1 = 1 = a_2 = a_3 = a_4 \quad \text{or} \quad a_1 = 2, a_2 = a_3 = a_4 = 0.$$

Suppose h is an odd number, then a_1 cannot be 2. Hence the equations, ($b > 0$, h odd)

$$h = \sum_{r=1}^{4} b_r, \qquad b = \sum_{r=1}^{4} b_r^2$$

are possible in integers b_r, if and only if $0 \leqslant 4b - h^2 \neq 4^\rho(8\sigma+7)$. This means that b must be odd, an obviously necessary condition since $b_r{}^2 - b_r$ is even. Hence, if $b > 0$ and b, h are odd numbers with $4b < h^2$, the equations always admit of integer solutions in b_r. They are well known for their connexion with Cauchy's proof of Fermat's theorem on polygonal numbers, where, however, the additional important restriction is required that $b_r \geqslant 0$.

Similarly, on taking $h = 0$, $a = 7, 11, 12, 14, 15$, we have the first five formulae in the scheme, in which the right-hand side denotes those numbers, and only those, which cannot be represented by the corresponding left-hand form:

$$\begin{array}{ll}
2x^2+2y^2+3z^2+2xy-2yz, & 4^\rho(8\sigma+1) \\
z^2+2x^2-2xy+6y^2, & 4^\rho(8\sigma+5) \\
z^2+2x^2+2xy+2y^2, & 4^\rho(8\sigma+5) \\
z^2+3x^2+2xy+5y^2, & 2^{2\rho+1}(8\sigma+1) \\
3z^2+2x^2+2xy+3y^2, & 4^\rho(8\sigma+1) \\
3z^2+2x^2+2xy+8y^2, & 3^{2\rho}(3\sigma+1).
\end{array}$$

The third is given by Dickson and the sixth is proved in § 6. We have for the proof of the third, $a = 12$,

$$a_1 = 2 = a_2 = a_3,\ a_4 = 0; \quad \text{or} \quad a_1 = 3,\ a_2 = 1 = a_3 = a_4.$$

The first case leads to

$$b = b_4^2 + b_1^2 + b_2^2 + (b_1 + b_2)^2,$$

or

$$b = b_4^2 + 2b_1^2 + 2b_1 b_2 + 2b_2^2.$$

The second gives

$$b = b_1^2 + b_2^2 + b_3^2 + (3b_1 + b_2 + b_3)^2$$

or writing

$$b_1 = x,\ b_2 = y - x,\ b_3 = z - x,$$

$$b = x^2 + (y-x)^2 + (z-x)^2 + (x+y+z)^2$$
$$= 4x^2 + 2y^2 + 2yz + 2z^2.$$

Both results are included in the result that, if $m > 0$,

$$m = x^2 + 2y^2 + 2yz + 2z^2$$

is impossible, if and only if $12m = 4^\rho(8\sigma + 7)$, i.e. if $m = 4^\rho(8\sigma + 5)$.

6. The results of § 3 are particular cases of a group of theorems. The essential points of the proof are

(A) the form $\sum_1^4 x_r^2$ is capable of composition with itself;

(B) it represents all integers $m > 0$;

(C) the number $\phi(m)$ of representations* of m is a function of m which satisfies $\phi(m)\phi(n) = \lambda\phi(mn)$ where λ is a constant provided m and n are subject to certain restrictions, e.g. $[m, n] = 1$ or 2 and $\lambda = 1$.

(D) simple conditions exist for the representation of m by $\sum_{r=2}^4 x_r^2$.

There are other quaternary forms having these properties. Thus if p, q are any constants,

$$f_1 = x_1^2 + py_1^2 + qz_1^2 + pqt_1^2 \tag{6}$$

has the property of (A). For, as is well known, $f = f_1 f_2$ where

$$\begin{aligned} x &= x_1x_2 - py_1y_2 - qz_1z_2 - pqt_1t_2, \\ y &= x_1y_2 + y_1x_2 + qz_1t_2 - qt_1z_2, \\ z &= x_1z_2 - py_1t_2 + z_1x_2 + pt_1y_2, \\ t &= x_1t_2 + y_1z_2 - z_1y_2 + t_1x_2. \end{aligned} \tag{6.1}$$

* Sometimes, as in my *Messenger* paper, $\phi(m)$ denotes the number of representations of $4m$; also restrictions may have to be imposed on x_r, e.g. x_r is odd.

It has the property (B) by a theorem of Liouville* in the seven cases $p=1$, $q=1, 2, 3$ and $p=2$, $q=2, 3, 4, 5$. The properties (C), (D) are also satisfied in a number of instances.

One illustration will suffice, say $p=1$, $q=3$. Then the number of representations† of an odd number m by the form (6) is four times the sum of those divisors of m prime to 3. Hence, if m, n are both odd and $[m, n]=1$,

$$f(m)f(n)=4f(mn).$$

On considering the linear substitution

$$\begin{pmatrix} x_1+\sqrt{(-p)}y_1, & \sqrt{q}z_1+\sqrt{(-pq)}t_1 \\ -\sqrt{q}z_1+\sqrt{(-pq)}t_1, & x_1-\sqrt{(-p)}y_1 \end{pmatrix},$$

it is easy to prove that, in (6.1), the same solution x, y, z, t occurs four times since there are only four integer solutions of

$$1=\xi^2+\eta^2+3\zeta^2+3\tau^2,$$

namely,

$$\xi=\pm 1,\ \eta=\zeta=\tau=0 \quad \text{and} \quad \eta=\pm 1,\ \xi=\zeta=\tau=0.$$

Since

$$\tfrac{1}{4}[4f(m)][4f(n)]=4f(mn),$$

this means that *all* the representations of mn by (6) are given by (6.1).

Next, from (5.13) $\quad \Delta=x^2+3y^2+3z^2$

is possible if, and only if, $0\leqslant\Delta\neq 3^{2\rho}(3\sigma+2)$. Hence, if a and b are odd and positive,

$$\left.\begin{aligned} a&=a_1^2+a_2^2+3a_3^2+3a_4^2,\\ h&=a_1b_1+a_2b_2+3a_3b_3+3a_4b_4,\\ b&=b_1^2+b_2^2+3b_3^2+3b_4^2 \end{aligned}\right\} \tag{6.2}$$

is possible in integers a_r, b_r, $(r=1,\ldots,4)$, i.e.

$$aX^2+2hXY+bY^2=(a_1X+b_1Y)^2+(a_2X+b_2Y)^2+$$
$$+3(a_3X+b_3Y)^2+3(a_4X+b_4Y)^2,$$

if and only if

$$0\leqslant\Delta\neq 3^{2\rho}(3\sigma+2).$$

The conditions a and b odd, $[a, b]=1$ are not necessary. For if $d=[a, h, b]$, d can be represented by (6) and the theorem is true if it holds for

$$\frac{a}{d}X^2+2\frac{h}{d}XY+\frac{b}{d}Y^2.$$

Hence we may take $[a, h, b]=1$ or 2. Then the condition that a, b are odd can be replaced by the one that $[a, h, b]\neq 2$.

* See Dickson, *History of the Theory of Numbers*, 3 (1923), 225–33, for various results on quaternary forms.

Hence the restriction on a, b in (6.2) is that for $\beta \geqslant 1$, a, h, b should not have 2^β, $2^{\beta-1}$, 2^β respectively for the highest powers of 2 dividing them, i.e. the form $aX^2+2hXY+bY^2$ must not be a multiple of an improperly primitive form.

For a few numerical illustrations take $h=0$, $a=5$. Then

$$a_1 = a_2 = a_3 = 1,\ a_4 = 0,$$

$$0 = b_1+b_2+3b_3,$$

$$b = b_1^2+b_2^2+3b_3^2+3b_4^2,$$

whence
$$b = (b_2+3b_3)^2+b_2^2+3b_3^2+3b_4^2,$$

and so
$$b = 2x^2+2xy+8y^2+3z^2$$

is solvable if, and only if, $5b \neq 3^{2\rho}(3\sigma+2)$, i.e. $b \neq 3^{2\rho}(3\sigma+1)$.

So $h=0$, $a=6$, $a_1=a_2=0$, $a_3=a_4=1$, $b_3+b_4=0$ gives (5.9).

Also $h=0$, $a=2$, $a_1=a_2=1$, $a_3=a_4=0$, $b_1+b_2=0$ gives (5.16).

Finally a few words as to what corresponds to § 4. We may take some multiple of a fifth square, e.g. once, twice, etc., and investigate the problem as in § 4. But I content myself with proposing the following problem. For what constant values of p_r, $r=1,\ldots,5$ can every positive definite quadratic form (1) be expressed in the form

$$aX^2+2hXY+bY^2 = \sum_{r=1}^{5} p_r(a_rX+b_rY)^2,$$

where a_r, b_r are integers?

ON THE RELATION BETWEEN CORRESPONDING PROBLEMS IN PLANE STRESS AND IN GENERALIZED PLANE STRESS

By L. N. G. FILON

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1. In Love's *Mathematical Theory of Elasticity*, § 146, is given the theory of two-dimensional generalized plane stress for an isotropic homogeneous perfectly elastic plane plate in equilibrium. Although the treatment there given omits consideration of body-force, it is easily generalized to include this case, as follows.

The faces of the plate are taken to be $z = \pm c$, and are assumed to be free from traction. The *actual* displacements and stresses will be denoted throughout by u, v, w, $\widehat{xx}$, $\widehat{yy}$, $\widehat{zz}$, $\widehat{yz}$, $\widehat{zx}$, $\widehat{xy}$, and the *mean* displacements and stresses in the plane of the plate (means being taken with regard to z) by U, V, $\breve{xx}$, $\breve{yy}$, $\breve{xy}$. As in Love's *Elasticity* the relations between the mean displacements and stresses are found, on the usual assumption of generalized plane stress, that the actual stress $\widehat{zz}$ is zero throughout the material, to be given by:

$$\breve{xx} = \lambda'(\partial U/\partial x + \partial V/\partial y) + 2\mu \partial U/\partial x, \tag{1.1}$$

$$\breve{yy} = \lambda'(\partial U/\partial x + \partial V/\partial y) + 2\mu \partial V/\partial y, \tag{1.2}$$

$$\breve{xy} = \mu(\partial U/\partial y + \partial V/\partial x), \tag{1.3}$$

where

$$\lambda' = 2\lambda\mu/(\lambda + 2\mu), \tag{2}$$

λ and μ being the usual Lamé constants.

The first two body-stress equations then give, when we take account of a body-force derived from a potential Ω,

$$\partial\widehat{xx}/\partial x + \partial\widehat{xy}/\partial y + \partial\widehat{xz}/\partial z = \rho\partial\Omega/\partial x,$$

$$\partial\widehat{xy}/\partial x + \partial\widehat{yy}/\partial y + \partial\widehat{yz}/\partial z = \rho\partial\Omega/\partial y,$$

and, averaging these equations over the thickness of the plate and using the fact that $\widehat{xz} = 0$, $\widehat{yz} = 0$ when $z = \pm c$, we find

$$\partial\breve{xx}/\partial x + \partial\breve{xy}/\partial y = \rho\partial\overline{\Omega}/\partial x, \tag{3.1}$$

$$\partial\breve{xy}/\partial x + \partial\breve{yy}/\partial y = \rho\partial\overline{\Omega}/\partial y, \tag{3.2}$$

so that the mean stresses can be derived from a stress-function χ of x, y, in the form

$$\breve{xx} = \rho\overline{\Omega} + \partial^2\chi/\partial y^2, \tag{4.1}$$

$$\breve{yy} = \rho\overline{\Omega} + \partial^2\chi/\partial x^2, \tag{4.2}$$

$$\breve{xy} = -\partial^2\chi/\partial x\partial y. \tag{4.3}$$

If we solve back (1.1), (1.2) for $\partial U/\partial x$, $\partial V/\partial y$, we find that

$$E\partial U/\partial x = \breve{xx} - \eta\breve{yy}, \tag{5.1}$$

$$E\partial V/\partial y = \breve{yy} - \eta\breve{xx}, \tag{5.2}$$

where E is Young's modulus and η Poisson's ratio.

Eliminating U, V between (5.1), (5.2), and (1.3), we obtain readily the consistency relation

$$(1+\eta)(\partial^2\breve{xx}/\partial x^2 + \partial^2\breve{yy}/\partial y^2 + 2\partial^2\breve{xy}/\partial x\partial y) = \nabla_1^2(\breve{xx}+\breve{yy}), \tag{6.1}$$

where ∇_1^2 stands for the operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

Substituting from equations (4) into the left-hand side of (6.1), we obtain the equation

$$(1+\eta)\rho\nabla_1^2\overline{\Omega} = \nabla_1^2(\breve{xx}+\breve{yy}), \tag{6.2}$$

and, completing the substitution on the right-hand side,

$$\nabla_1^2\{\nabla_1^2\chi + \rho(1-\eta)\overline{\Omega}_1\} = 0, \tag{6.3}$$

which is the differential equation satisfied by the stress-function in this case.

Note carefully that, since the plate is assumed homogeneous, the density ρ, as well as all the elastic constants, is independent of x, y, z.

2. It is also known (see Love's *Theory of Elasticity*, §§ 145, 302) that in such a plate a stress-system can exist, such that, not only $\widehat{zz}$, but $\widehat{xz}$ and $\widehat{yz}$ vanish throughout the body of the plate, the only non-vanishing stresses being those in the plane of the plate. Such a system is spoken of as one of plane stress. If, further, the displacements u, v are even functions of z, and w an odd function of z, the middle surface of the plate is distorted entirely in its own plane. The stresses $\widehat{xx}$, $\widehat{yy}$, $\widehat{xy}$ are then even functions of z. This may be referred to briefly as symmetrical plane stress.

It is clear that symmetrical plane stress satisfies the conditions required for generalized plane stress. In fact, it more than satisfies them, since it implies the vanishing of the shears $\widehat{xz}$, $\widehat{yz}$, which is not necessarily required for generalized plane stress.

Thus to every symmetrical plane stress there corresponds a generalized plane stress. The object of the present note is to show

that, subject to certain conditions to be satisfied by the potential of the body-forces, the converse holds good, namely, that to every generalized plane stress there corresponds a uniquely determined symmetrical plane stress, and to obtain, in a simple form, the displacements and stresses in this symmetrical plane stress directly from the mean displacements and mean stresses in the corresponding generalized plane stress.

In §§ 15–17 some examples are given of the application of this procedure to certain important cases of flexure of vertically flat girders and cantilevers.

3. It is to be noted, in the first place, that, if such a correspondence is to exist, the potential Ω of the body-forces must be independent of z, and not merely symmetrical in z, for no plane-stress solution (for equilibrium) can exist unless $\partial\Omega/\partial z = 0$.

This follows at once from the third body-stress equation

$$\partial\widehat{xz}/\partial x + \partial\widehat{yz}/\partial y + \partial\widehat{zz}/\partial z = \rho\partial\Omega/\partial z,$$

since for plane stress $\widehat{xz} = \widehat{yz} = \widehat{zz} = 0$.

We may now omit the bar over the Ω in § 1 and treat Ω as a function of x, y only.

4. From the usual stress-strain relations

$$\widehat{xx} = \lambda(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + 2\mu\partial u/\partial x, \tag{7.1}$$

$$\widehat{yy} = \lambda(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + 2\mu\partial v/\partial y, \tag{7.2}$$

$$\widehat{zz} = \lambda(\partial u/\partial x + \partial v/\partial y + \partial w/\partial z) + 2\mu\partial w/\partial z = 0, \tag{7.3}$$

we find easily, as in generalized plane stress,

$$\widehat{xx} = \lambda'(\partial u/\partial x + \partial v/\partial y) + 2\mu\partial u/\partial x, \tag{8.1}$$

$$\widehat{yy} = \lambda'(\partial u/\partial x + \partial v/\partial y) + 2\mu\partial v/\partial y. \tag{8.2}$$

To these last equations we may add

$$\widehat{xy} = \mu(\partial u/\partial y + \partial v/\partial x). \tag{8.3}$$

We have also

$$E\partial u/\partial x = \widehat{xx} - \eta\widehat{yy}, \tag{9.1}$$

$$E\partial v/\partial y = \widehat{yy} - \eta\widehat{xx}, \tag{9.2}$$

$$E\partial w/\partial z = -\eta(\widehat{xx} + \widehat{yy}). \tag{9.3}$$

All the above are identical with the corresponding equations of generalized plane stress.

5. Further, the body-stress equations give (since $\widehat{xz} = \widehat{yz} = \widehat{zz} = 0$)

$$\partial\widehat{xx}/\partial x + \partial\widehat{xy}/\partial y = \rho\partial\Omega/\partial x, \tag{10.1}$$

$$\partial\widehat{xy}/\partial x + \partial\widehat{yy}/\partial y = \rho\partial\Omega/\partial y, \tag{10.2}$$

whence the stresses may be expressed in terms of a stress-function χ_1 as follows:

$$\widehat{xx} = \rho\Omega + \partial^2\chi_1/\partial y^2, \tag{11.1}$$

$$\widehat{yy} = \rho\Omega + \partial^2\chi_1/\partial x^2, \tag{11.2}$$

$$\widehat{xy} = -\partial^2\chi_1/\partial x\partial y. \tag{11.3}$$

These have the same form as in generalized plane stress, but now χ_1 is a function, not of x, y only, but of x, y, and z.

The equations of § 4 clearly lead to consistency relations of the same form as (6.1), (6.2), and (6.3), namely,

$$(1+\eta)(\partial^2\widehat{xx}/\partial x^2 + \partial^2\widehat{yy}/\partial y^2 + 2\partial^2\widehat{xy}/\partial x\partial y) = \nabla_1^2(\widehat{xx}+\widehat{yy}), \tag{12.1}$$

$$(1+\eta)\rho\nabla_1^2\Omega = \nabla_1^2(\widehat{xx}+\widehat{yy}), \tag{12.2}$$

$$\nabla_1^2\{\nabla_1^2\chi_1 + \rho\Omega(1-\eta)\} = 0. \tag{13}$$

6. If we now subtract corresponding equations in the two systems, we obtain

$$\widehat{xx} - \breve{xx} = \lambda'\{\partial(u-U)/\partial x + \partial(v-V)/\partial y\} + 2\mu\partial(u-U)/\partial x, \tag{14.1}$$

$$\widehat{yy} - \breve{yy} = \lambda'\{\partial(u-U)/\partial x + \partial(v-V)/\partial y\} + 2\mu\partial(v-V)/\partial y, \tag{14.2}$$

$$E\partial(u-U)/\partial x = \widehat{xx} - \breve{xx} - \eta(\widehat{yy} - \breve{yy}), \tag{14.3}$$

$$E\partial(v-V)/\partial y = \widehat{yy} - \breve{yy} - \eta(\widehat{xx} - \breve{xx}), \tag{14.4}$$

$$\widehat{xy} - \breve{xy} = \mu\{\partial(u-U)/\partial y + \partial(v-V)/\partial x\}, \tag{14.5}$$

$$\widehat{xx} - \breve{xx} = \partial^2(\chi_1 - \chi)/\partial y^2, \tag{14.6}$$

$$\widehat{yy} - \breve{yy} = \partial^2(\chi_1 - \chi)/\partial x^2, \tag{14.7}$$

$$\widehat{xy} - \breve{xy} = -\partial^2(\chi_1 - \chi)/\partial x\partial y, \tag{14.8}$$

where

$$\nabla_1^4(\chi_1 - \chi) = 0. \tag{14.9}$$

7. So far, we have not taken account of the conditions satisfied for plane stress

$$\widehat{xz} = 0, \qquad \widehat{yz} = 0,$$

which lead to

$$\partial u/\partial z = -\partial w/\partial x, \qquad \partial v/\partial z = -\partial w/\partial y,$$

and these may be written, since $\partial U/\partial z = 0$, $\partial V/\partial z = 0$,

$$\partial(u-U)/\partial z = -\partial w/\partial x, \tag{15.1}$$

$$\partial(v-V)/\partial z = -\partial w/\partial y. \tag{15.2}$$

Differentiating with regard to z, we have

$$\partial^2(u-U)/\partial z^2 = -(\partial/\partial x)(\partial w/\partial z), \tag{16.1}$$

$$\partial^2(v-V)/\partial z^2 = -(\partial/\partial y)(\partial w/\partial z), \tag{16.2}$$

whence, using (9.3),

$$\partial^2(u-U)/\partial z^2 = (\eta/E)(\partial/\partial x)(\widehat{xx}+\widehat{yy}), \tag{17.1}$$

$$\partial^2(v-V)/\partial z^2 = (\eta/E)(\partial/\partial y)(\widehat{xx}+\widehat{yy}). \tag{17.2}$$

Differentiating (17.1) with regard to x, (17.2) with regard to y, and adding

$$(\partial^2/\partial z^2)\{\partial(u-U)/\partial x+\partial(v-V)/\partial y\}=(\eta/E)\nabla_1^2(\widehat{xx}+\widehat{yy}), \tag{18}$$

whence, comparing (18) and (12.2), we find

$$\begin{aligned}(\partial^2/\partial z^2)\{\partial(u-U)/\partial x+\partial(v-V)/\partial y\} &= \{\eta(1+\eta)\rho/E\}\nabla_1^2\Omega \\ &= (\eta\rho/2\mu)\nabla_1^2\Omega.\end{aligned} \tag{19}$$

Integrate (19) twice with regard to z, and note, (i) that the term in z must be absent since u, v are to be even in z, (ii) that the mean value of $\partial(u-U)/\partial x+\partial(v-V)/\partial y$ is zero; we have

$$\partial(u-U)/\partial x+\partial(v-V)/\partial y=(\eta\rho/4\mu)(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega, \tag{20}$$

or, using (14.3) and (14.4),

$$(1-\eta)(\widehat{xx}+\widehat{yy}-\breve{xx}-\breve{yy})=\tfrac{1}{2}\eta\rho(1+\eta)(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega. \tag{21}$$

Hence, from (14.6) and (14.7),

$$\nabla_1^2(\chi_1-\chi)=\{\rho\eta(1+\eta)/2(1-\eta)\}(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega. \tag{22}$$

8. Comparison of equations (22) and (14.9) shows at once that, for a plane-stress solution to exist, we must have

$$\nabla_1^4\Omega=0. \tag{23}$$

This, however, leads to no real limitation if the field of body-force is purely gravitational, for, since Ω is strictly two-dimensional, $\nabla_1^2\Omega=-4\gamma\pi\rho$ by Poisson's equation, and ρ being constant, $\nabla_1^2\rho=0$ and therefore $\nabla_1^4\Omega=0$.

If, however, the field of force were due to a distribution of electric charges, (23) might lead to an effective limitation.

The complete limitation on Ω turns out, however, to be narrower than is implied by equation (23).

Differentiating (14.3) twice with regard to z and using (17.1), we obtain

$$\eta(\partial^2/\partial x^2)(\widehat{xx}+\widehat{yy})=(\partial^2/\partial z^2)(\widehat{xx}+\widehat{yy}-\breve{xx}-\breve{yy})-(1+\eta)(\partial^2/\partial z^2)(\widehat{yy}-\breve{yy}),$$

and, using (21) and (14.7),

$$\begin{aligned}(\partial^2/\partial x^2)\{\eta(\widehat{xx}+\widehat{yy})+(1+\eta)(\partial^2/\partial z^2)(\chi_1-\chi)\}& \\ =\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega.&\end{aligned} \tag{24.1}$$

Similarly, from (14.4),

$$\begin{aligned}(\partial^2/\partial y^2)\{\eta(\widehat{xx}+\widehat{yy})+(1+\eta)(\partial^2/\partial z^2)(\chi_1-\chi)\}& \\ =\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega.&\end{aligned} \tag{24.2}$$

Differentiating likewise (14.5) twice with regard to z and using (17.1), (17.2), and (14.8), we find, on rearranging,

$$(\partial^2/\partial x\partial y)\{\eta(\widehat{xx}+\widehat{yy})+(1+\eta)(\partial^2/\partial z^2)(\chi_1-\chi)\}=0. \tag{24.3}$$

To discuss these equations, write for shortness

$$\eta(\widehat{xx}+\widehat{yy})+(1+\eta)(\partial^2/\partial z^2)(\chi_1-\chi)=R. \tag{25}$$

Then equations (24) become

$$\left.\begin{aligned}\partial^2R/\partial x^2=\partial^2R/\partial y^2=\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega\\ \partial^2R/\partial x\partial y=0.\end{aligned}\right\} \tag{26}$$

From the last of equations (26), treating for the moment x, y as the only variables, we get

$$R=F(x)+G(y).$$

The first and second of (26) then give

$$F''(x)=G''(y)=\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega,$$

whence clearly $F''(x)$ and $G''(y)$ must each be independent of x, y; hence $\nabla_1^2\Omega$ is independent of x, y, and, since it is otherwise independent of z,

$$\nabla_1^2\Omega=\text{const.} \tag{27}$$

As explained previously, this condition will always be satisfied if Ω is a gravitational potential, in virtue of Poisson's equation.

9. If we now substitute into (17.1) and (17.2) from (21) for $\widehat{xx}+\widehat{yy}$ and remember (27), we have

$$2\mu(\partial^2/\partial z^2)(u-U)=\{\eta/(1+\eta)\}(\partial/\partial x)(\breve{xx}+\breve{yy}),$$

$$2\mu(\partial^2/\partial z^2)(v-V)=\{\eta/(1+\eta)\}(\partial/\partial y)(\breve{xx}+\breve{yy}).$$

Integrate these twice with regard to z, omitting odd powers of z and adjusting the arbitrary functions of x, y so as to make the mean values of $u-U$, $v-V$ zero:

$$2\mu(u-U)=\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\partial/\partial x)(\breve{xx}+\breve{yy}), \tag{28.1}$$

$$2\mu(v-V)=\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\partial/\partial y)(\breve{xx}+\breve{yy}). \tag{28.2}$$

These give the required corrections to the mean displacements.

10. To find the stresses, substitute into equations (14). The procedure is simplified if we write (14.3), (14.4) in the form

$$\widehat{xx}-\breve{xx}=2\mu\partial(u-U)/\partial x+\{\eta/(1+\eta)\}(\widehat{xx}+\widehat{yy}-\breve{xx}-\breve{yy}),$$

$$\widehat{yy}-\breve{yy}=2\mu\partial(v-V)/\partial y+\{\eta/(1+\eta)\}(\widehat{xx}+\widehat{yy}-\breve{xx}-\breve{yy}),$$

and now substitute on the right-hand sides from (28.1), (28.2), and (21). The result gives

$$\widehat{xx}-\breve{xx} = \{\eta^2\rho/2(1-\eta)\}(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega+ \\ +\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\partial^2/\partial x^2)(\breve{xx}+\breve{yy}), \qquad (29.1)$$

$$\widehat{yy}-\breve{yy} = \{\eta^2\rho/2(1-\eta)\}(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega+ \\ +\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\partial^2/\partial y^2)(\breve{xx}+\breve{yy}). \qquad (29.2)$$

For the shear $\widehat{xy}$ substitute from (28.1), (28.2) into (14.5) direct and we have

$$\widehat{xy}-\breve{xy} = \{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\partial^2/\partial x\partial y)(\breve{xx}+\breve{yy}). \qquad (29.3)$$

These give the required corrections to the mean stresses.

11. To find the transverse displacement w, we substitute from (28.1) and (28.2) into (15.1) and (15.2).

$$-2\mu\partial w/\partial x = \{\eta z/(1+\eta)\}(\partial/\partial x)(\breve{xx}+\breve{yy}),$$

$$-2\mu\partial w/\partial y = \{\eta z/(1+\eta)\}(\partial/\partial y)(\breve{xx}+\breve{yy}),$$

whence

$$-2\mu w = \{\eta z/(1+\eta)\}(\breve{xx}+\breve{yy})+H(z).$$

To determine $H(z)$ we use equation (9.3). Thus

$$\eta(\widehat{xx}+\widehat{yy}) = \eta(\breve{xx}+\breve{yy})+(1+\eta)H'(z),$$

which, from (21), leads to

$$\{\eta^2\rho(1+\eta)/2(1-\eta)\}(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega = (1+\eta)H'(z).$$

Integrating this, and remembering that w, and therefore $H(z)$, must be odd in z, we have

$$H(z) = \{\eta^2\rho/6(1-\eta)\}(z^3-c^2z)\nabla_1^2\Omega,$$

so that

$$2\mu w = -\{\eta^2\rho/6(1-\eta)\}z(z^2-c^2)\nabla_1^2\Omega-\{\eta z/(1+\eta)\}(\breve{xx}+\breve{yy}), \qquad (30)$$

which gives the third displacement and completes the solution.

12. It is desirable to obtain explicitly the function χ_1. Referring back to equations (26), and having regard to (27), they clearly lead to

$$R = \tfrac{1}{2}(x^2+y^2)\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega+\text{arbitrary function of } z. \qquad (31)$$

(25) then gives us

$$(1+\eta)(\partial^2/\partial z^2)(\chi_1-\chi) = R-\eta(\widehat{xx}+\widehat{yy}-\breve{xx}-\breve{yy})-\eta(\breve{xx}+\breve{yy}) \\ = \tfrac{1}{2}(x^2+y^2)\{\eta\rho(1+\eta)/(1-\eta)\}\nabla_1^2\Omega-\eta(\breve{xx}+\breve{yy})+ \\ +\text{arbitrary function of } z,$$

the middle term on the right-hand side having been absorbed into the arbitrary function, since (21) and (27) together show that the term in question is a function of z only.

Integrating now the equation last written, and using the condition that $\chi_1 - \chi$ is to be an even function of z whose mean value is zero, we obtain

$$\chi_1 - \chi = \tfrac{1}{4}(x^2+y^2)\{\eta\rho/(1-\eta)\}(z^2-\tfrac{1}{3}c^2)\nabla_1^2\Omega - \\ -\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\breve{xx}+\breve{yy}), \quad (32)$$

the arbitrary function of z being finally omitted, since it clearly would contribute nothing to the stresses and may therefore be dropped without affecting the solution in any way.

It is easily verified that, if we substitute from (32) into (14.6), (14.7), and (14.8), bearing in mind (6.2), these equations are satisfied. The solution is therefore consistent, since all the equations (14) have been shown to be satisfied, and it is clearly uniquely determinate.

13. We have therefore shown that, provided the force-potential Ω is such that $\nabla_1^2\Omega = \text{const.}$, there exists always a symmetrical plane stress corresponding to a given generalized plane stress, and that the actual displacements and stresses in this symmetrical plane stress are derived directly from the mean displacements and mean stresses by adding corrective terms, as given by equations (28) and (29); and these corrective terms are known when the single quantity $\breve{xx}+\breve{yy}$ has been determined in the generalized plane stress, and do not depend directly on the boundary conditions.

Moreover, if we refer to (30), we see that

$$2\mu(w_{+c}-w_{-c}) = -\{2\eta c/(1+\eta)\}(\breve{xx}+\breve{yy}),$$

so that $\breve{xx}+\breve{yy} = (E/2\eta c)\times$(lateral contraction of the plate at the given point).

In photo-elastic investigations of stresses in plates considerable use is made (e.g. by Professor Coker) of such measurements of contraction or expansion by means of a lateral extensometer. The data given by such measurements, provided a formula can be fitted to them reasonably well, so as to enable the differentiations in equations (29) to be carried out, will enable the experimenter to estimate the departure of the actual stress in the plate from the average, a point of some importance.

14. The most frequent and useful case occurs when Ω is the gravitational potential of masses at a distance, usually the earth. In this case $\nabla_1^2\Omega = 0$, and, if we write

$$\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(\breve{xx}+\breve{yy}) = \phi, \quad (33)$$

the solution assumes the extremely simple form

$$\chi_1 = \chi - \phi, \tag{34.1}$$

$$2\mu(u-U, v-V, -w) = (\partial/\partial x, \partial/\partial y, \partial/\partial z)\phi, \tag{34.2}$$

$$(\widehat{xx}-\breve{xx}, \widehat{xy}-\breve{xy}, \widehat{yy}-\breve{yy}) = (\partial^2/\partial x^2, \partial^2/\partial x\partial y, \partial^2/\partial y^2)\phi. \tag{34.3}$$

15. We will now take a few examples of the applications of these results.

Consider the generalized plane stress given by

$$\chi = My^3/6I, \qquad \Omega = 0,$$

I being the moment of inertia of the rectangle $y = \pm b$, $z = \pm c$ about the axis of z.

This corresponds to

$$\breve{xx} = My/I, \qquad \breve{yy} = \breve{xy} = 0,$$

and therefore to a flat girder of thickness $2c$ and height $2b$, bent in its own plane under a uniform bending-moment M.

It is then found easily by the usual methods that

$$U = Mxy/EI, \qquad V = -(M/2EI)(x^2+\eta y^2).$$

We have now

$$\phi = \{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)(My/I),$$

whence

$$\widehat{xx} = \breve{xx} = My/I,$$

$$\widehat{xy} = \breve{xy} = 0,$$

$$\widehat{yy} = \breve{yy} = 0,$$

$$u = U+(1/2\mu)\partial\phi/\partial x = U = Mxy/EI,$$

$$v = V+(1/2\mu)\partial\phi/\partial y = -(M/2EI)\{x^2+\eta(y^2-z^2+\tfrac{1}{3}c^2)\},$$

$$w = -(1/2\mu)\partial\phi/\partial z = -(M/EI)yz.$$

These are the well-known displacements of Euler-Bernoulli flexure, and it is at once seen that, in this case, they apply to a beam of any cross-section, since the stress across any element parallel to the axis of x vanishes.

16. Take next the case where

$$\chi = (W/2I)(\tfrac{1}{3}xy^3-b^2xy), \qquad \Omega = 0,$$

the symbols having the same meaning as in the last section.

This leads to

$$\breve{xx} = Wxy/I, \qquad \breve{yy} = 0, \qquad \breve{xy} = (W/2I)(b^2-y^2),$$

corresponding to a flat cantilever, whose plane is vertical, and which carries a total transverse load W at $x=0$.

Here $$\phi=\{\eta/2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)Wxy/I,$$

and
$$\left.\begin{aligned}\widehat{xx}=\breve{xx},\qquad \widehat{yy}&=\breve{yy},\\ \widehat{xy}=\breve{xy}+\{\eta/2(1+\eta)\}&(z^2-\tfrac{1}{3}c^2)W/I\\ &=(W/2I)[b^2-y^2+\{\eta/(1+\eta)\}(z^2-\tfrac{1}{3}c^2)].\end{aligned}\right\} \qquad (35)$$

In this case the top and bottom faces are under a residual traction

$$\widehat{xy}=\{W\eta/2I(1+\eta)\}(z^2-\tfrac{1}{3}c^2),$$

which is, however, self-equilibrating at every cross-section, so that, applying de Saint-Venant's principle, the stresses inside the body of the plate should be given to a sufficient approximation by (35). If, in fact, the well-known solution for flexure with shear of such a beam of rectangular cross-section be examined, and terms which become exponentially small at a distance from the top and bottom be neglected, the approximate stresses deduced are found to be identical with those given by (35).

The displacements in this case also are readily calculated as before, if required. The only laborious part is the preliminary calculation of U and V, but the algebra is quite straightforward.

17. Finally, to illustrate the case where there is a body-force, we will consider a problem for which

$$\chi=Ax^2y+Bx^2y^3+Cy^5+Dy^3,\qquad \Omega=gy.$$

Since $\nabla_1^4\chi=0$ (for $\nabla_1^2\Omega=0$), we must have $C=-B/5$, so that

$$\chi=Ax^2y+B(x^2y^3-\tfrac{1}{5}y^5)+Dy^3.$$

Choose now A and B so that the mean stresses $\breve{xy}$, $\breve{yy}$ both vanish over $y=\pm b$. This requires

$$\partial^2\chi/\partial x^2=-\rho gy \text{ and } \partial^2\chi/\partial x\partial y=0$$

when $y=\pm b$, leading to:

$$A+Bb^2=-\tfrac{1}{2}\rho g,\qquad A+3Bb^2=0,$$

or $A=-\tfrac{3}{4}\rho g$, $B=\tfrac{1}{4}(\rho g/b^2)$, so that

$$\chi=\tfrac{1}{4}(\rho g/b^2)(x^2y^3-3x^2yb^2-\tfrac{1}{5}y^5)+Dy^3.$$

This gives the generalized plane stress for a heavy uniform girder of density ρ, height $2b$, and thickness $2c$, under its own weight.

The mean stresses are given by

$$\breve{xx} = (\rho gy/2b^2)(2b^2+3x^2-2y^2)+6Dy,$$
$$\breve{yy} = (\rho gy/2b^2)(y^2-b^2),$$
$$\breve{xy} = (3\rho gx/2b^2)(b^2-y^2).$$

To find the total transverse shear S per unit thickness and the bending-moment M per unit thickness, we have

$$S = \int_{-b}^{b} \breve{xy}\,dy = 2\rho gbx,$$

$$M = -\int_{-b}^{b} \breve{xx}\,y\,dy = -\rho gb(x^2+\tfrac{4}{15}b^2)-4Db^3.$$

If the girder is simply supported at its ends, which are distant a from the plane $x = 0$ of symmetry, then

$$4Db^3 = -\rho gb(a^2+\tfrac{4}{15}b^2),$$

and $M = \rho gb(a^2-x^2)$.

The stress-function then becomes

$$\chi = (\rho g/4b^2)(x^2y^3-a^2y^3-3x^2yb^2-\tfrac{1}{5}y^5-\tfrac{4}{15}b^2y^3)$$

and $$\breve{xx} = (\rho gy/2b^2)\{3(x^2-a^2)-2y^2+\tfrac{6}{5}b^2\}.$$

Hence $$\breve{xx}+\breve{yy} = (\rho gy/2b^2)\{3(x^2-a^2)-y^2+\tfrac{1}{5}b^2\},$$

and $$\phi = \{\eta\rho gy/4b^2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)\{3(x^2-a^2)-y^2+\tfrac{1}{5}b^2\},$$

so that, in the corresponding plane stress,

$$\widehat{xx} = \breve{xx}+\{3\eta\rho gy/2b^2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)$$
$$= (\rho gy/2b^2)[3(x^2-a^2)-2y^2+\tfrac{6}{5}b^2+\{3\eta/(1+\eta)\}(z^2-\tfrac{1}{3}c^2)],$$
$$\widehat{yy} = \breve{yy}-\{3\eta\rho gy/2b^2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)$$
$$= (\rho gy/2b^2)[y^2-b^2-\{3\eta/(1+\eta)\}(z^2-\tfrac{1}{3}c^2)],$$
$$\widehat{xy} = \breve{xy}+\{3\eta\rho gx/2b^2(1+\eta)\}(z^2-\tfrac{1}{3}c^2)$$
$$= (3\rho gx/2b^2)[b^2-y^2+\{\eta/(1+\eta)\}(z^2-\tfrac{1}{3}c^2)].$$

These should give close approximations to the actual stress at any point in such a girder at a distance from the top and bottom faces. In this case, however, no exact solution for a heavy rectangular beam is available for comparison.

The displacements, in this case also, are not difficult to obtain in finite terms, but they lead to long algebraic expressions and it appears hardly worth while to write them down.

THE EXPONENTIAL FUNCTION IN LINEAR ALGEBRAS

By P. DIENES

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1. The introduction of complex numbers was chiefly suggested by the problem of determining the zeros of polynomials. Some integral functions, such as e^x, have no zero in the field of complex numbers. This fact suggests the following question. Can we generalize the idea of number to such an extent that the exponential function may have a zero in the extended field?

We shall prove in this Note that the *exponential function has no zero in the linear associative algebras to a finite base*, and that it has no self-associative zero in finite non-associative linear algebras. This result extends to a large class of algebras to an infinite base. In particular, the exponential function has no zero in the tensor algebra of relativity theory and it misses only singular tensor values which do not divide some tensor. Moreover, in Hilbert's* algebra of infinite bounded matrices, so important in atom mechanics, the exponential function has no absolutely bounded matrix zero.

Since we have to define the symbol

$$1+\sum_{n=1}^{\infty} x^n/n!$$

for numbers x of a general linear algebra, it is necessary to suppose that 1 is a number of the algebra, i.e. that the algebra has a modulus e_0 such that if x is any number of the algebra, $xe_0 = e_0x = x$. Obviously the algebra must also contain a zero element.

2. Construction of an algebra to an infinite base

The numbers of an algebra to the base $e_0 = 1, e_1, \ldots, e_n, \ldots$ are, by definition, the forms $x = \sum_{i=0}^{\infty} x_i e_i$, where the components x_i are real or complex numbers. Addition is defined as addition of the corresponding components. Multiplication is supposed to be distributive with respect to addition and is defined (a) by the multiplication table

$$e_i e_j = \sum_{k=0}^{\infty} m_{ijk} e_k, \tag{1}$$

* D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Leipzig (1912), pp. 128–9.

where m_{ijk} are arbitrarily given real or complex numbers, i.e. by

$$xy = \sum_{k=0}^{\infty}\left(\sum_{i,j=0}^{\infty} x_i y_j m_{ijk}\right) e_k, \tag{2}$$

and (b) by the convention that if c is any real or complex number,

$$cx = xc = \sum_{i=0}^{\infty} cx_i e_i. \tag{3}$$

Zero is the number of the algebra whose components are all zero. We see that $0x = x0 = 0$.

We also notice that multiplication does not lead to a number, unless the infinity of double series

$$\sum_{i,j=0}^{\infty} x_i y_j m_{ijk} \ (k = 0, 1, \ldots)$$

are all convergent, so that, in every algebra to an infinite base, there are numbers whose product is meaningless, unless, for every k and for $i > N$, $j > N$, we have $m_{ijk} = 0$. In fact, we can choose x_i, y_i such that $x_i y_j m_{ijk} = 1$ for an infinity of suffixes i and j.

The necessary and sufficient condition that the multiplication of units should be associative is that $(e_i e_j)e_k = e_i(e_j e_k)$, i.e.

$$\sum_{\alpha=0}^{\infty} m_{ij\alpha} m_{\alpha kl} = \sum_{\alpha=0}^{\infty} m_{i\alpha l} m_{jk\alpha} \tag{4}$$

for every i, j, k, l. In an algebra to an infinite base these conditions are not sufficient to make multiplication associative in the whole algebra. In fact, the detailed form of the equation $(xy)z = x(yz)$ is

$$\sum_{\alpha k} \sum_{ij} x_i y_j z_k m_{ij\alpha} m_{\alpha kl} = \sum_{\alpha i} \sum_{jk} x_i y_j z_k m_{jk\alpha} m_{i\alpha l}, \tag{5}$$

and unless the order of summation can be inverted, (5) is not necessarily a consequence of (4).

For example, in the infinite matrix algebra to a double array of units e_{ij} with the multiplication table

$$e_{ij} e_{kl} = \begin{cases} 0 & \text{if } j \neq k \\ e_{il} & \text{if } j = k, \end{cases} \tag{6}$$

multiplication of units is associative, but for suitably chosen coefficients, the equation

$$\left(\sum a_{ij} e_{ij} \sum b_{kl} e_{kl}\right) \sum c_{mn} e_{mn} = \sum a_{ij} e_{ij} \left(\sum b_{kl} e_{kl} \sum c_{mn} e_{mn}\right)$$

i.e.
$$\sum_{\beta}\left(\sum_{\alpha} a_{i\alpha} b_{\alpha\beta}\right) c_{\beta n} = \sum_{\alpha} a_{i\alpha} \sum_{\beta} b_{\alpha\beta} c_{\beta n}$$

is not satisfied. This remark seems to be necessary, for in papers on quantum theory the associative property is taken for granted also for non-bounded infinite matrices.

3. Power series

We say that a power series $\sum_{n=0}^{\infty} a_n x^n$, where the coefficients a_n and the variable x belong to a given algebra, is convergent if all the products $a_n x^n = \sum_i (a_n x^n)_i e_i$ have a sense and if all the ordinary infinite series $\sum_n (a_n x^n)_i$, $(i = 0, 1, \ldots \infty)$, are convergent. We readily see that, if c is any real or complex number,

$$[a_n (cx)^n]_i = c^n (a_n x^n)_i$$

as soon as $(a_n x^n)_i$ exists. Hence:

THEOREM I. *If $\sum a_n x^n$ is convergent, $\sum a_n (cx)^n$ is absolutely convergent provided the absolute value of the complex number c is less than* 1.

We shall now examine the Cauchy product of two infinite series. If the product

$$\Big(\sum_n a_n \sum_n b_n\Big)_k = \sum_{ij} m_{ijk} \sum_n (a_n)_i \sum_n (b_n)_j \tag{1}$$

exists, we have

$$\Big(\sum_n a_n \sum_n b_n\Big)_k = \sum_{ij} m_{ijk} \sum_n [(a_0)_i (b_n)_j + (a_1)_i (b_{n-1})_j + \ldots + (a_n)_i (b_0)_j], \tag{2}$$

provided the rearrangement is legitimate, e.g. when the two series $\sum_n (a_n)_i$ and $\sum_n (b_n)_j$ are absolutely convergent (or at least one of them). If also the order of the two successive summations, viz. with respect to n, and with respect to i, j, can be inverted, we obtain

$$\Big(\sum_n a_n \sum_n b_n\Big)_k = \sum_n [a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0]_k \tag{3}$$

$$\sum_n a_n \sum_n b_n = \sum_n [a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0] \tag{4}$$

which corresponds to a well-known theorem on infinite series with real or complex terms.

The last inversion is obviously allowed if i and j assume only a finite number m of integral values, i.e. the base of the algebra is finite. Hence:

THEOREM II. *If, in an algebra to a finite base, both $\sum a_n$ and $\sum b_n$ are absolutely convergent, the Cauchy product of the two series is absolutely convergent and we have*

$$\sum a_n \sum b_n = \sum (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0). \tag{5}$$

4. The exponential function in finite linear algebras

We say that x is self-associative if

$$x^{n'}x^{p'} = x^n x^p \tag{1}$$

whenever $n'+p' = n+p$. Take a self-associative number x of a finite algebra and consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{i=0}^{m-1} \left[\sum_{n=0}^{\infty} \frac{1}{n!}(x^n)_i \right] e_i \tag{2}$$

Denoting by M the largest of the numbers $|m_{ijk}|$ and by ρ the largest of $|x_i|$, we have

$$|(x^n)_i| \leqslant \rho(m^2\rho M)^{n-1}, \tag{3}$$

which establishes the absolute convergence of all the m series of the right-hand side of (2) and thus proves

THEOREM I. *In a finite linear algebra the right-hand side of*

$$E(x) = 1 + \sum x^n/n! \tag{4}$$

defines the exponential function for every self-associative number x of the algebra. In a finite associative algebra, $E(x)$ is defined by (4) *over the whole algebra.*

The exponential function so defined is continuous if we define continuity by the condition that $f(x^{(n)}) \to f(x)$ for every sequence $x^{(n)} \to x$ [i.e. $\lim_{n\to\infty}(x^{(n)})_i = x_i$ for every i]. In fact, since (2) consists of a finite number of infinite series, uniform convergence in any finite domain of the algebra is established by (3) and, as usual, continuity follows from uniform convergence.

The direct result of applying Theorem II, § 3, to the exponential function is

$$E(rx)E(\bar{r}x) = E(rx+\bar{r}x), \qquad r, \bar{r} \text{ real or complex}, \tag{5}$$

provided x is self-associative. A particular case of (5) is

$$E(-x)E(x) = E(0) = 1. \tag{6}$$

But $E(-x)$ is an ordinary (non-infinite number) of our algebra. Therefore, if we suppose that $E(x) = 0$, (6) leads to an absurdity. Finally x is certainly self-associative in associative algebras. Thus we have

THEOREM II. *The exponential function has no zero in finite linear associative algebras. The exponential function has no self-associative zero in finite linear algebras.*

The question arises, 'Does the exponential function miss values

other than zero?' Theorem II, § 3, applied to $\sum x^n/n!$ and $\sum \bar{x}^n/n!$, where x and $\bar{x}$ are commutable, i.e. $x\bar{x} = \bar{x}x$, easily leads to the equations

$$E(x)E(\bar{x}) = E(x+\bar{x}), \text{ if } x\bar{x} = \bar{x}x, \tag{7}$$

$$E(nx)E(px) = E(rx) = E(n'x)E(p'x), \text{ if } n+p = n'+p' = r. \tag{8}$$

It follows that if x is self-associative, then also $E(x)$ is self-associative. Also, if $E(x) = a$, then $a^p = E(px)$, i.e. if a is a value of $E(x)$, then all the positive powers of a are assumed by $E(x)$. Hence:

THEOREM III. *All the values of $E(x)$ assumed for self-associative x are self-associative. The exponential function misses all the nil-potent numbers of every finite linear associative algebra. The exponential function misses all the self-associative nil-potent numbers of every finite linear algebra.*

The equation

$$E(x) = E\left(\frac{x}{q}+\frac{x}{q}+\dots+\frac{x}{q}\right) = \left[E\left(\frac{x}{q}\right)\right]^q$$

leads to the definition of qth root. If a is a value of $E(x)$, $a = E(\bar{x})$ say, we put

$$a^{1/q} = E(\bar{x}/q), \qquad a^{p/q} = E(p\bar{x}/q), \tag{9}$$

which, by the continuity of $E(x)$, leads to the definition of a real (or complex) power r of a,

$$a^r = E(r\bar{x}). \tag{10}$$

The power so defined possesses the two fundamental properties of indices, viz.

$$a^r a^s = a^{r+s}, \qquad (a^r)^s = a^{rs}. \tag{11}$$

If $b = E(\bar{y})$ and $\bar{x}$, $\bar{y}$ are commutable, in which case a and b are themselves commutable, we also have

$$a^r b^r = (ab)^r. \tag{12}$$

The extension of the results of this article to algebras to an infinite base is a delicate problem. We shall discuss the exponential function in two different types of infinite algebras used in relativity theory and atom mechanics respectively.

5. The exponential function in tensor algebra

Consider an algebra to an infinite base where the multiplication table is replaced by the convention that all the products

$$e_i e_j, \qquad (e_i e_j)e_k = e_i(e_j e_k) = e_i e_j e_k, \text{ etc.}$$

of the given set of 'fundamental units' $e_1, e_2,\dots, e_m$ are fresh units

of the algebra. Thus, besides $a_0+\sum_{i=1}^{m} a_i e_i$, the algebra contains numbers of the form

$$\sum_{i,j=1}^{m} a_{ij}e_ie_j,\ldots, \sum_{i,\ldots,i_k=1}^{m} a_{i_1\ldots i_k}e_{i_1}\ldots e_{i_k},\ldots,$$

where e_ie_j and e_je_i are different units. In this section we shall make use of the summation convention, i.e. we shall suppress $\sum$ if the same suffix occurs twice in a product. Thus $\sum_{i,j=1}^{m} a_{ij}e_ie_j$ will be denoted by $a_{ij}e_ie_j$. The general form of a number of this algebra is an infinite polynomial in $e_1,\ldots,e_n$ with real or complex coefficients:

$$A = a_0+a_ie_i+a_{ij}e_ie_j+\ldots+a_{i_1\ldots i_k}e_{i_1}\ldots e_{i_k}+\ldots$$
$$B = b_0+b_ie_i+b_{ij}e_ie_j+\ldots+b_{i_1\ldots i_k}e_{i_1}\ldots e_{i_k}+\ldots$$

Sum and product are defined by putting

$$A+B = a_0+b_0+(a_i+b_i)e_i+(a_{ij}+b_{ij})e_ie_j+\ldots+ \\ +(a_{i_1\ldots i_k}+b_{i_1\ldots i_k})e_{i_1}\ldots e_{i_k}+\ldots \quad (1)$$

$$AB = a_0b_0+(a_0b_i+a_ib_0)e_i+(a_0b_{ij}+a_ib_j+a_{ij}b_0)e_ie_j+\ldots+ \\ +(a_0b_{i_1\ldots i_k}+a_{i_1}b_{i_2\ldots i_k}+\ldots+a_{i_1\ldots i_k}b_0)e_{i_1}\ldots e_{i_k}+\ldots \quad (2)$$

We notice* that the product always exists, that $0A = A0=0$, and that if $AB = 0$ then either A or B (or both) are 0.

The algebra so constructed will be called the *tensor algebra* to the fundamental base $e_1,\ldots,e_m$, for the tensor algebra of relativity theory is of this type, usually restricted, however, to homogeneous forms.

THEOREM I. *If, in a linear associative algebra, $xy = yx$ and n is a positive integer, then*

$$(x+y)^n = x^n+c_1^n x^{n-1}y+\ldots+c_k^n x^{n-k}y^k+\ldots+y^n. \quad (3)$$

Proof. Multiplying $xy = yx$ by x and y on the right and left we see that $x^2y = xyx = yx^2$, $xy^2 = y^2x$. In general $x^my^k = y^kx^m$. Therefore any proof of the binomial theorem for real or complex x and y extends without the slightest modification to any pair of commutable numbers x and y of a linear associative algebra.

THEOREM II. *The exponential function*

$$E(x) = \sum x^n/n! \quad (4)$$

has a definite tensor value for every tensor x.

* See P. Dienes, 'Sur la structure mathématique du Calcul Tensoriel': *Journal de Math.* Ser. 9. vol. 3 (1924), pp. 79–106.

Proof. Putting $x = x_0 + X, \quad X = x_i e_i + x_{ij} e_i e_j + \ldots$ we have, by (3),

$$x^n = x_0^n + n x_0^{n-1} X + \frac{n(n-1)}{1.2} x_0^{n-2} X^2 + \ldots + X^n,$$

for x_0 is commutable with every tensor.

Putting
$$X^2 = X^2_{ij} e_i e_j + \ldots + X^2_{i_1 \ldots i_r} e_{i_1} \ldots e_{i_r} + \ldots$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$X^s = X^s_{i_1 \ldots i_s} e_{i_1} \ldots e_{i_s} + \ldots \tag{5}$$

we see that the coefficient of $e_{i_1} \ldots e_{i_k}$ in $x^n/n!$ is

$$\frac{x_0^{n-1}}{(n-1)!} x_{i_1 \ldots i_k} + \frac{x_0^{n-2}}{(n-2)!} \frac{X^2_{i_1 \ldots i_k}}{2!} + \ldots + \frac{x_0^{n-k}}{(n-k)!} \frac{X^k_{i_1 \ldots i_k}}{k!}, \tag{6}$$

i.e. it consists of k terms (when $n \geqslant k$), the summation with respect to n affecting only x_0. Therefore, when we sum with respect to n, the coefficient of $e_{i_1} \ldots e_{i_k}$ in $\sum x^n/n!$ will be the finite sum

$$e^{x_0} x_{i_1 \ldots i_k} + e^{x_0} \frac{X^2_{i_1 \ldots i_k}}{2!} + \ldots + e^{x_0} \frac{X^k_{i_1 \ldots i_n}}{k!} = \{E(x)\}_{i_1 \ldots i_k} \tag{7}$$

which proves the theorem.

In particular, the terms without $e_1, \ldots, e_m$ in $\sum x^n/n!$ lead to e^{x_0} as the 0th coordinate of $E(x)$. But $E(x) = 0$ means that every coordinate of $E(x)$ is zero. Since the 0th coordinate cannot be zero, we have

THEOREM III. *The exponential function has no tensor zero.*

We also notice that

$$E(x+y) = E(x)E(y), \text{ if } xy = yx.$$

Left-hand division of A by B requires the solution of the equation

$$Bx = A,$$

i.e. the equations

$$b_0 x_0 = a_0$$
$$b_i x_0 + b_0 x_i = a_i$$
$$b_{ij} x_0 + b_i x_j + b_0 x_{ij} = a_{ij}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$b_{i_1 \ldots i_k} x_0 + b_{i_1 \ldots i_{k-1}} x_{i_k} + \ldots + b_0 x_{i_1 \ldots i_k} = a_{i_1 \ldots i_k}.$$

Therefore, if $b_0 \neq 0$, left-hand (and right-hand) division by B is possible and the result is unique. If $b_0 = 0$ and $a_0 \neq 0$, the division of A by B is impossible.

We see thus that if $b_0 = 0$, there are tensors A such that the division of A by B is not possible (B is a singular tensor), and if

$b_0 \neq 0$, division of A by B is possible for every A (B is a regular tensor). The exponential function assumes only regular tensor values.

On the other hand, it assumes all such values. In fact equating (7) to any given $a_{i_1 \dots i_k}$, the values x_0, $x_i, x_{ij}, \dots, x_{i_1 \dots i_k}$ satisfying those equations are successively determined by the equations themselves since $X^2_{i_1 \dots i_k} \dots X^k_{i_1 \dots i_k}$ contains only $x_i, x_{ij}, \dots, x_{i_1 \dots i_{k-1}}$ and the coefficient of $x_{i_1 \dots i_k}$ is $e^{x_0} \neq 0$. Therefore

THEOREM IV. *The exponential function assumes every regular tensor and misses all singular tensors.*

6. The exponential function in matrix algebra

Matrix algebra is specified by the multiplication table (2.6). We say, with Hilbert, that the real numbers x_i form a *unit vector* if $\sum_{i=0}^{\infty} x_i^2 = 1$ and that the matrix $a_{ij} e_{ij}$ is *bounded* if

$$\left| \sum_{ij=0}^{\infty} a_{ij} x_i x_j \right| \leqslant M \quad \text{for every unit vector.} \tag{1}$$

The least number M satisfying (1) is the *bound* of the matrix.

Hilbert proved (*a*) that the product matrix $a_{ij} b_{jk} e_{ik}$ is bounded by the product MN of their respective bounds; (*b*) that multiplication of bounded matrices is associative.

It follows from (1) that

$$|a_{ij}| \leqslant M, \tag{2}$$

i.e. the components of a matrix are bounded by the bound of the matrix (but the converse is not true). The two properties of multiplication established by Hilbert show that every power x^n of any bounded matrix x bounded by M is a perfectly determinate bounded matrix bounded by M^n, i.e. by (2),

$$|(x^n)_{\alpha\beta}| \leqslant M^n. \tag{3}$$

We thus see that the series $1 + \sum x^n/n!$, where 1 is the unit matrix, converges for every bounded matrix x. Therefore

THEOREM I. *The exponential function is defined for the whole algebra of bounded matrices by*

$$E(x) = 1 + \sum x^n/n! \tag{4}$$

and all its values are bounded matrices.

Now consider two series

$$\sum a_n \text{ and } \sum b_n \tag{5}$$

of absolutely bounded matrices, i.e. such that $A_{ijn} = |(a_n)_{ij}|$ and $B_{ijn} = |(b_n)_{ij}|$ are also bounded matrices bounded by A_n and B_n respectively. If we suppose that both series (5) are absolutely convergent, we have

$$(\sum a_n \sum b_n)_{ik} = \sum_j \sum_n (a_n)_{ij} \sum_n (b_n)_{jk}$$
$$= \sum_j \sum_n [(a_0)_{ij}(b_n)_{jk} + (a_1)_{ij}(b_{n-1})_{jk} + \dots + (a_n)_{ij}(b_0)_{jk}].$$

Since, by hypothesis, a series remains convergent when we replace a_n and b_n by A_n and B_n, the series converges absolutely, and thus, inverting the order of summation, we obtain

$$\sum a_n \sum b_n = \sum_n [a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0]. \tag{6}$$

Since bounded matrices are associative, they satisfy (4.1), and thus, by the repetition of the reasoning used in the case of finite algebras, we obtain $E(-x)E(x) = 1$ leading to

THEOREM II. *The exponential function has no absolutely bounded matrix zero.*

7. The exponential function in bounded algebras

The main difficulty in examining the exponential function by our method in infinite algebras consists in the inversion of the order of summation on the right-hand side of (3.2). We can justify this inversion by a convenient restriction of the multiplication parameters m_{ijk} or by restricting the series $\sum a_n$, $\sum b_n$, i.e., in the case of power series, the values of x, or, again, we might make use of all these restrictions.

As an example we shall consider 'bounded' algebras, i.e. such that

$$\sum_{i,j} |m_{ijk}| = M_k \leqslant M, \tag{1}$$

and we shall restrict x and the coefficients a_n, b_n to 'bounded numbers' of the algebra in question, viz. such that

$$|x_i| \leqslant \rho, \qquad |(a_n)_i| \leqslant A_n, \qquad |(b_n)_i| \leqslant B_n. \tag{2}$$

It follows that
$$|(a_n x^n)_i| \leqslant A_n \rho^n M^n, \tag{3}$$

so that, if
$$\overline{\lim_{n=\infty}} |\sqrt[n]{A_n}| = \frac{1}{R}, \tag{4}$$

the series $\sum a_n x^n$ will be absolutely convergent if $M\rho < R$. The value R/M is a kind of radius of convergence for $\sum a_n x^n$. If R' is the

corresponding number for $\sum b_n x^n$ and $\bar{R}$ is the lesser of the two, the series

$$\sum_n \rho^n (A_0 B_n + A_1 B_{n-1} + \ldots + A_n B_n) = C(\rho) \tag{5}$$

converges if $\rho < \bar{R}$.

On the other hand, if x is self-associative, commutable with all the coefficients a_n and b_n, and if $|x_i| \leqslant \rho$, we have

$$|(\sum_n a_n x^n \sum_n b_n x^n)_k|$$
$$= |\sum_{ij} m_{ijk} \sum_n [(a_0)_i (b_n x^n)_j + (a_1 x)_i (b_{n-1} x^{n-1})_j + \ldots + (a_n x^n)_i (b_0)_j]|$$
$$\leqslant \sum_{ij} |m_{ijk}| \sum_n (A_0 B_n + A_1 B_{n-1} + \ldots + A_n B_0) \rho^n < MC(\rho),$$

by (5) and (1). The triple series being absolutely convergent, we can rearrange it at will. Thus we obtain

THEOREM I. *If x is a self-associative number of a bounded algebra, commutable with all the coefficients a_n and b_n, and if R and R' are the radii of convergence of $\sum a_n x^n$ and $\sum b_n x^n$ respectively, and $\bar{R}$ is the lesser of the two, the Cauchy product $\sum (a_0 b_n + a_1 b_{n-1} + \ldots + a_n b_0) x^n$ exists for every x such that $|x_i| \leqslant \rho < \bar{R}$ and is equal to $\sum a_n x^n \sum b_n x^n$. The product so obtained is a bounded number.*

The direct result of applying Theorem I to the exponential series, where the coefficients are positive and R infinite, is

THEOREM II. *The exponential function is defined for every self-associative bounded number of a bounded algebra, all its values so defined are self-associative bounded numbers and it misses all the self-associative nil-potent bounded numbers of the algebra.*

The last part of the theorem follows, as usual, from $E(rx)E(\bar{r}x) = E(rx + \bar{r}x)$, showing that with a every power of a is assumed by $E(x)$.

For associative bounded algebras the last result reduces to the simpler form:

THEOREM III. *In associative bounded algebras the exponential function exists for every bounded number, and all these values of $E(x)$ are bounded numbers. $E(x)$ has no bounded zero and misses all the nil-potent bounded numbers of the algebra.*

THE CONSTANTS OF LANDAU AND LEBESGUE

By G. N. WATSON

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1. It has been proved by Landau* that, if $f(z)$ is a function which is analytic throughout the interior of the circle $|z| = 1$ and which is expansible in the Taylor series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots,$$

and if $|f(z)| < 1$ whenever $|z| < 1$, then

$$|a_0 + a_1 + a_2 + \ldots + a_n| \leqslant G_n,$$

where

$$G_n = 1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1.3}{2.4}\right)^2 + \ldots + \left(\frac{1.3.5\ldots(2n-1)}{2.4.6\ldots\ldots.2n}\right)^2.$$

Landau has also shown that, corresponding to any given value of n, a function can be constructed such that $|a_0 + a_1 + a_2 + \ldots + a_n|$ is actually equal to G_n.

It is evident from Cauchy's theorem that

$$a_0 + a_1 + a_2 + \ldots + a_n = \frac{1}{2\pi i} \int_C f(z) \left\{ \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots + \frac{1}{z^{n+1}} \right\} dz,$$

where the contour C is the circle $|z| = r < 1$.

Hence

$$|a_0 + a_1 + a_2 + \ldots + a_n| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(re^{i\theta}) \frac{r^{n+1} e^{(n+1)i\theta} - 1}{r^n e^{ni\theta}(re^{i\theta} - 1)} \, d\theta \right|$$

$$\leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| \left| \frac{r^{n+1} e^{(n+1)i\theta} - 1}{r^n e^{ni\theta}(re^{i\theta} - 1)} \right| d\theta$$

$$< \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{r^{n+1} e^{(n+1)i\theta} - 1}{r^n e^{ni\theta}(re^{i\theta} - 1)} \right| d\theta.$$

* *Archiv der Math. und Phys.* (3) 21 (1913), 42–50, 250–5; *Ergebnisse der Funktionentheorie* (1929), 26–9.

Since this inequality holds for all positive values of r less than 1, we must have

$$|a_0+a_1+a_2+\ldots+a_n| \leqslant \lim_{r\to 1-0} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{r^{n+1}e^{(n+1)i\theta}-1}{r^n e^{ni\theta}(re^{i\theta}-1)} \right| d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin \frac{1}{2}(n+1)\theta}{\sin \frac{1}{2}\theta} \right| d\theta$$

$$= \frac{1}{\pi} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \left| \frac{\sin(n+1)\theta}{\sin\theta} \right| d\theta$$

$$= \frac{2}{\pi} \int_{0}^{\frac{1}{2}\pi} \frac{|\sin(n+1)\theta|}{\sin\theta}\, d\theta = \rho_{\frac{1}{2}n},$$

where ρ_n denotes Lebesgue's constant* defined by the equation

$$\rho_n = \frac{2}{\pi} \int_{0}^{\frac{1}{2}\pi} \frac{|\sin(2n+1)\theta|}{\sin\theta}\, d\theta.$$

When we give a_0, a_1, a_2, ..., a_n the special values whose sum is numerically equal to G_n, it is evident from these inequalities that†

$$G_n \leqslant \rho_{\frac{1}{2}n}. \tag{1}$$

By taking $n = 1$ in this inequality, we infer that $\pi < 3{\cdot}2$.

Analysis of the type which has just been given shows that, if $F(\theta)$ is any integrable function such that $|F(\theta)| \leqslant 1$, then

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta) \frac{e^{(n+1)i\theta}-1}{e^{ni\theta}(e^{i\theta}-1)}\, d\theta \right| \leqslant \rho_{\frac{1}{2}n};$$

and it is fairly evident that, corresponding to any given value of n, the bound $\rho_{\frac{1}{2}n}$ can be attained by choosing the function $F(\theta)$ in an appropriate manner.

The last integral, however, has an upper bound G_n which is less than $\rho_{\frac{1}{2}n}$ when $F(\theta)$ is the limit, as $r \to 1$, of an analytic function $f(re^{i\theta})$. It is consequently of some little interest to investigate the relative magnitudes of G_n and $\rho_{\frac{1}{2}n}$ as an estimate of the effect of prescribing that $F(\theta)$ should be the limit of an analytic function.

* *Leçons sur les séries trigonométriques* (1906), 86–7. In work on Fourier series it is adequate to consider ρ_n where n is an integer; but the properties of $\rho_{\frac{1}{2}n}$ which will be required here are similar to those of ρ_n.

† So far as I know, the inequality (1) has not previously been noticed.

It is evident that $G_0 = \rho_0 = 1$; and, when n is large,

$$G_n \sim \frac{1}{\pi}\log n, \qquad \rho_{\frac{1}{2}n} = \frac{4}{\pi^2}\log n + c + o(1),$$

where c is independent of n; these formulae are due to Landau and Fejér* respectively.

The results which have just been stated indicate the possibility of the truth of the inequality

$$G_n \leqslant \rho_{\frac{1}{2}n} < \frac{4}{\pi} G_n \qquad (n = 0, 1, 2, \ldots), \tag{2}$$

part of which has already been proved, and further that, when n assumes integral values, $\rho_{\frac{1}{2}n}/G_n$ may be an increasing function of n, so that

$$1 \leqslant \frac{\rho_{\frac{1}{2}n}}{G_n} < \frac{\rho_{\frac{1}{2}(n+1)}}{G_{n+1}} < \frac{4}{\pi} \qquad (n = 0, 1, 2, \ldots). \tag{3}$$

2. The conjectures (2) and (3) prove to be correct, and they will be established in §§ 5, 6; before investigating them, it is convenient to make a digression by examining the asymptotic expansions of $\rho_{\frac{1}{2}n}$ and G_n, which, I think, have not been published previously.

Take Szegö's formula† for Lebesgue's constant

$$\rho_{\frac{1}{2}n} = \frac{16}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{4m^2-1}\left(\frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{2m(n+1)-1}\right); \tag{4}$$

if $\psi(x)$ denotes the logarithmic derivate of the gamma function, we have

$$\begin{aligned}\rho_{\frac{1}{2}n} &= \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{4m^2-1}\{\psi(mn+m+\tfrac{1}{2}) - \psi(\tfrac{1}{2})\} \\ &= -\frac{4}{\pi^2}\psi(\tfrac{1}{2}) + \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\psi(mn+m+\frac{1}{2})}{4m^2-1}.\end{aligned}$$

Now, in the asymptotic expansion

$$\psi(mn+m+\tfrac{1}{2}) \sim \log(mn+m) + \sum_{r=1}^{\infty}\frac{(-)^{r-1}B_r}{2r}\left(1 - \frac{1}{2^{2r-1}}\right)\frac{1}{m^{2r}(n+1)^{2r}},$$

the error due to stopping at any term is of the same sign as, and is

* *Journal für Math.* 138 (1910), 22–53.

† *Math. Zeitschrift*, 9 (1921), 163–6. It is worth mentioning that this was obtained very ingeniously by expanding $|\sin\phi|$ in a Fourier cosine-series, deducing that

$$|\sin\phi| = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{\sin^2 m\phi}{4m^2-1},$$

writing $(n+1)\theta$ for ϕ and integrating term by term.

numerically less than, the first term neglected; and, since $(4m^2-1)^{-1}$ is positive, it is easy to see that when we substitute the asymptotic expansion of $\psi(mn+m+\frac{1}{2})$ in (4), we obtain the asymptotic expansion of $\rho_{\frac{1}{2}n}$ with the same properties for the error. That is to say,

$$\rho_{\frac{1}{2}n} \sim \frac{4}{\pi^2}\Big[\log(n+1)+\Big\{2\sum_{m=1}^{\infty}\frac{\log m}{4m^2-1}-\psi(\tfrac{1}{2})\Big\}+$$
$$+\sum_{r=1}^{\infty}\frac{(-)^{r-1}B_r}{r}\Big(1-\frac{1}{2^{2r-1}}\Big)\Big\{\sum_{m=1}^{\infty}\frac{1}{(4m^2-1)m^{2r}}\Big\}\frac{1}{(n+1)^{2r}}\Big],$$

so that

$$\rho_{\frac{1}{2}n} \sim \frac{4}{\pi^2}\Big[\log(n+1)+c_0+\sum_{r=1}^{\infty}\frac{(-)^{r-1}c_r}{(n+1)^{2r}}\Big], \tag{5}$$

where

$$c_0 = 2\sum_{m=1}^{\infty}\frac{\log m}{4m^2-1}-\psi(\tfrac{1}{2}), \tag{6}$$

$$c_r = \frac{B_r}{r}\Big(1-\frac{1}{2^{2r-1}}\Big)\sum_{m=1}^{\infty}\frac{1}{(4m^2-1)m^{2r}} \quad (r = 1, 2, 3,\ldots). \tag{7}$$

The constants c_r $(r \geqslant 1)$ are expressible in finite terms; thus

$$\frac{1}{2^{2r}}\sum_{m=1}^{\infty}\frac{1}{(4m^2-1)m^{2r}} = \sum_{m=1}^{\infty}\Big\{\frac{1}{4m^2-1}-\frac{1}{(2m)^2}-\frac{1}{(2m)^4}-\cdots-\frac{1}{(2m)^{2r}}\Big\}$$
$$= \frac{1}{2}\Big[1-\sum_{s=1}^{r}\frac{B_s\pi^{2s}}{(2s)!}\Big],$$

so that

$$c_r = \frac{(2^{2r-1}-1)B_r}{r}\Big[1-\sum_{m=1}^{r}\frac{B_m\pi^{2m}}{(2m)!}\Big] \quad (r = 1, 2, 3,\ldots). \tag{7a}$$

The constant c_0 is expressible in a form suitable for computation as follows:

$$2\sum_{m=1}^{\infty}\frac{\log m}{4m^2-1} = \sum_{m=1}^{\infty}\Big(\frac{\log m}{2m-1}-\frac{\log m}{2m+1}\Big)$$
$$= \sum_{m=1}^{\infty}\Big(\frac{\log(m+1)}{2m+1}-\frac{\log m}{2m+1}\Big) = \sum_{m=1}^{\infty}\frac{1}{2m+1}\log\frac{(2m+1)+1}{(2m+1)-1}$$
$$= 2\sum_{r=0}^{\infty}\frac{1}{2r+1}\sum_{m=1}^{\infty}\frac{1}{(2m+1)^{2r+2}} = 2\sum_{r=0}^{\infty}\frac{U_{2r+2}-1}{2r+1},$$

where $U_{2r+2} = \sum_{m=0}^{\infty}\frac{1}{(2m+1)^{2r+2}}$.

The values of U_{2r+2} have been computed by Glaisher,* and $U_{2r+2}-1$ decreases sufficiently rapidly, as r increases, to make the computation of $2\sum_{r=0}^{\infty}\frac{U_{2r+2}-1}{2r+1}$ an easy matter.

We thus get

$$\begin{array}{l|l} c_0 = 2{\cdot}44132\ 38136\ 94835, & c_4 = 0{\cdot}00275\ 82678\ 40582, \\ c_1 = 0{\cdot}02958\ 88277\ 62644, & c_5 = 0{\cdot}00504\ 16328\ 46519, \\ c_2 = 0{\cdot}00492\ 82989\ 42233, & c_6 = 0{\cdot}01405\ 56865\ 05171, \\ c_3 = 0{\cdot}00257\ 11773\ 16860, & c_7 = 0{\cdot}05554\ 94529\ 62768. \end{array}$$

I have calculated $\rho_{\frac{1}{2}n}$ when $n = 9$ by means of this asymptotic expansion and also by direct calculation from Lebesgue's integral, using Andoyer's fifteen-figure tables of the circular functions. The results agreed to fifteen significant figures, giving

$$\rho_{9/2} = 1{\cdot}92275\ 35823\ 6529.$$

3. The asymptotic expansion of G_n is a little more troublesome to investigate. Take Ramanujan's formula†

$$G_n = \left\{\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\right\}^2 \left\{\frac{1}{n+1} + \left(\frac{1}{2}\right)^2 \frac{1}{n+2} + \left(\frac{1.3}{2.4}\right)^2 \frac{1}{n+3} + \ldots\right\},$$

whence it follows that

$$\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2 G_n = 2\int_0^1 \left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \ldots\right\} k^{2n+1}\,dk$$

$$= \frac{4}{\pi}\int_0^1 K k^{2n+1}\,dk = \frac{4}{\pi}\int_0^1 K'(1-k^2)^n k\,dk$$

$$= \frac{2}{\pi^2}\int_0^1 \sum_{m=0}^{\infty} \left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\right\}^2 k^{2m+1}(1-k^2)^n \times$$

$$\times\{-\log k^2 + 2\psi(m+1) - 2\psi(m+\tfrac{1}{2})\}\,dk$$

$$= \frac{1}{\pi^2}\sum_{m=0}^{\infty} \left\{\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)}\right\}^2 \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)} \times$$

$$\times\{\psi(m+n+2) + \psi(m+1) - 2\psi(m+\tfrac{1}{2})\}.$$

Since $$\sum_{m=0}^{\infty} \frac{\{\Gamma(m+\frac{1}{2})\}^2\Gamma(n+1)}{\Gamma(m+1)\Gamma(m+n+2)} = \pi\left\{\frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\right\}^2,$$

* *Quart. J. of Math.* 45 (1914), 141–58.

† For proofs of this formula, see Watson, *Journal London Math. Soc.* 4 (1929), 82–6; Darling, ibid. 5 (1930), 8–9.

we deduce that

$$G_n = \frac{1}{\pi}\log(n+1) + \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\pi^2\Gamma(n+1)} \sum_{m=0}^{\infty} \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)\Gamma(m+n+2)} \times$$
$$\times\{\psi(m+n+2)-\log(n+1)+\psi(m+1)-2\psi(m+\tfrac{1}{2})\}.$$

Now let M be any fixed integer, not less than 2; it is easy to verify that, when $m \geqslant M$,

$$0 < \psi(m+n+2)+\psi(m+1) < 2m+n+3 \leqslant 2(m+n+1),$$

and

$$0 < \log(n+1)+2\psi(m+\tfrac{1}{2}) < (n+1)+2(m+1) \leqslant 2(m+n+1).$$

Hence

$$\left|\frac{\{\Gamma(n+\frac{3}{2})\}^2}{\pi^2\Gamma(n+1)} \sum_{m=M}^{\infty} \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)\Gamma(m+n+2)} \times \right.$$
$$\left. \times\{\psi(m+n+2)-\log(n+1)+\psi(m+1)-2\psi(m+\tfrac{1}{2})\}\right|$$
$$< \frac{2\{\Gamma(n+\frac{3}{2})\}^2}{\pi^2\Gamma(n+1)} \sum_{m=M}^{\infty} \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)\Gamma(m+n+1)}$$
$$< \frac{2\{\Gamma(n+\frac{3}{2})\}^2}{\pi^2\Gamma(n+1)(M+2)(M+3)\ldots(M+n)} \sum_{m=M}^{\infty} \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)\Gamma(m+2)}$$
$$< \frac{2\{\Gamma(n+\frac{3}{2})\}^2\Gamma(M+2)}{\pi^2\Gamma(n+1)\Gamma(M+n+1)} F(\tfrac{1}{2},\tfrac{1}{2};2;1) = O\left(\frac{1}{(n+1)^{M-1}}\right)$$

when n is large, since M is fixed. Consequently

$$G_n = \frac{1}{\pi}\log(n+1) + \frac{\{\Gamma(n+\frac{3}{2})\}^2}{\pi^2\Gamma(n+1)} \sum_{m=0}^{M-1} \frac{\{\Gamma(m+\frac{1}{2})\}^2}{\Gamma(m+1)\Gamma(m+n+2)} \times$$
$$\times\{\psi(m+n+2)-\log(n+1)+\psi(m+1)-2\psi(m+\tfrac{1}{2})\}+O\{(n+1)^{1-M}\},$$

and the asymptotic expansion of G_n up to the term containing $(n+1)^{2-M}$ is obtainable by substituting the asymptotic expansions of the functions which occur in this terminating series.

The asymptotic expansion, as far as I have calculated it, is

$$G_n \sim \frac{1}{\pi}\left[\log(n+1)+\gamma+4\log 2-\frac{1}{4(n+1)}+\frac{5}{192(n+1)^2}+\ldots\right], \quad (8)$$

and $\gamma+4\log 2 = 3{\cdot}34980\ 43871$.

4. We now revert to the problem of proving (2) and (3). As a preliminary we shall show that

$$(2n+2)^2(\rho_{\frac{1}{2}(n+1)}-\rho_{\frac{1}{2}n})-(2n+1)^2(\rho_{\frac{1}{2}n}-\rho_{\frac{1}{2}(n-1)}) > 0 \quad (n = 1,2,3,\ldots). \quad (9)$$

First observe that, when $n = 1$, the expression on the left in (9) is equal to

$$\frac{43\pi - 300 + 96\sqrt{3}}{3\pi},$$

and this is positive. It is consequently sufficient to discuss (9) when n assumes the values 2, 3, 4,....

It follows from (4) that

$$\rho_{\frac{1}{2}n} = \frac{16}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{4m^2-1} \int_0^{\infty} (e^{-t} + e^{-3t} + \ldots + e^{-\{2m(n+1)-1\}t})\, dt$$

$$= \frac{16}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{4m^2-1} \int_0^{\infty} \frac{e^{-t} - e^{-\{2m(n+1)+1\}t}}{1-e^{-2t}}\, dt,$$

so that

$$\rho_{\frac{1}{2}n} - \rho_{\frac{1}{2}(n-1)} = \frac{16}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{4m^2-1} \int_0^{\infty} \frac{\sinh mt}{\sinh t}\, e^{-m(2n+1)t}\, dt,$$

and hence

$$(2n+2)^2(\rho_{\frac{1}{2}(n+1)} - \rho_{\frac{1}{2}n}) - (2n+1)^2(\rho_{\frac{1}{2}n} - \rho_{\frac{1}{2}(n-1)})$$

$$= \frac{16}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{4m^2-1} \int_0^{\infty} \frac{\sinh mt}{\sinh t} \{(2n+2)^2 e^{-m(2n+3)t} - (2n+1)^2 e^{-m(2n+1)t}\}\, dt.$$

It is evident from this result that, in order to prove (9) when $n \geqslant 2$, it is adequate to prove that

$$\int_0^{\infty} \frac{\sinh mt}{\sinh t} \{(2n+2)^2 e^{-m(2n+3)t} - (2n+1)^2 e^{-m(2n+1)t}\}\, dt > 0 \tag{10}$$

$$(n \geqslant 2;\, m = 1, 2, 3, \ldots).$$

Now consider (10); for brevity write

$$\{(2n+2)^2 e^{-m(2n+3)t} - (2n+1)^2 e^{-m(2n+1)t}\} \cosh(m-1)t \equiv \phi(t),$$

$$\frac{\sinh mt}{\sinh t \cosh(m-1)t} \equiv \chi(t).$$

For any given values of m and n, it is obvious that $\phi(t)$ vanishes for only one positive value of t, say $t = \alpha$; and

$$\phi(t) > 0, \quad (t < \alpha); \qquad \phi(t) < 0, \quad (t > \alpha).$$

It is evident that $\chi(t)$ is a constant when $m = 1$ or $m = 2$, and it is easy to prove that it is a decreasing function of t when $m > 2$; for

$$\chi'(t) = \frac{(m-1)\sinh 2t - \sinh(2m-2)t}{2\sinh^2 t \cosh^2(m-1)t} < 0,$$

when $m-1>1$, as may be seen by expanding the numerator in powers of t. Hence

$$\frac{\chi(t)}{\chi(\alpha)} \geqslant 1 \quad (t \leqslant \alpha); \qquad 0 \leqslant \frac{\chi(t)}{\chi(\alpha)} \leqslant 1 \quad (t \geqslant \alpha). \tag{11}$$

Now consider $\int_0^\infty \phi(t)\,dt$; we have

$$\int_0^\infty \phi(t)\,dt = \frac{(2n+2)^2}{2}\left\{\frac{1}{m(2n+3)-(m-1)}+\frac{1}{m(2n+3)+(m-1)}\right\} -$$

$$-\frac{(2n+1)^2}{2}\left\{\frac{1}{m(2n+1)-(m-1)}+\frac{1}{m(2n+1)+(m-1)}\right\}$$

$$= \frac{m(2n+2)^2(2n+3)}{m^2(2n+3)^2-(m-1)^2} - \frac{m(2n+1)^3}{m^2(2n+1)^2-(m-1)^2}$$

$$= \frac{m\{m^2(2n+1)^2(2n+3)-(m-1)^2(16n^2+26n+11)\}}{\{m^2(2n+3)^2-(m-1)^2\}\{m^2(2n+1)^2-(m-1)^2\}}$$

$$> \frac{m(m-1)^2\{(2n+1)^2(2n+3)-(16n^2+26n+11)\}}{\{m^2(2n+3)^2-(m-1)^2\}\{m^2(2n+1)^2-(m-1)^2\}}$$

$$= \frac{4m(m-1)^2(n+1)(2n^2-n-2)}{\{m^2(2n+3)^2-(m-1)^2\}\{m^2(2n+1)^2-(m-1)^2\}}$$

$$\geqslant 0 \text{ when } n \geqslant 2.$$

Hence, when $n \geqslant 2$, $\quad \int_0^\alpha \phi(t)\,dt > \int_\alpha^\infty \{-\phi(t)\}\,dt$

and *a fortiori*, by (11),

$$\int_0^\alpha \phi(t)\frac{\chi(t)}{\chi(\alpha)}\,dt > \int_\alpha^\infty \{-\phi(t)\}\frac{\chi(t)}{\chi(\alpha)}\,dt,$$

that is to say, $$\int_0^\infty \phi(t)\chi(t)\,dt > 0.$$

Hence (9) is true when $n \geqslant 2$, and it has already been proved when $n=1$; it is consequently true for all of the specified values of n.

5. It follows at once from (9) that, when $N > n \geqslant 1$,

$$\frac{\rho_{\frac{1}{2}n}-\rho_{\frac{1}{2}(n-1)}}{\left\{\dfrac{1.3...(2n-1)}{2.4........2n}\right\}^2} < \frac{\rho_{\frac{1}{2}(n+1)}-\rho_{\frac{1}{2}n}}{\left\{\dfrac{1.3...(2n+1)}{2.4...(2n+2)}\right\}^2} < \dots$$

$$< \frac{\rho_{\frac{1}{2}N}-\rho_{\frac{1}{2}(N-1)}}{\left\{\dfrac{1.3...(2N-1)}{2.4........2N}\right\}^2} < \lim_{N\to\infty} \frac{\rho_{\frac{1}{2}N}-\rho_{\frac{1}{2}(N-1)}}{\left\{\dfrac{1.3...(2N-1)}{2.4........2N}\right\}^2} = \frac{4}{\pi},$$

so that

$$\rho_{\frac{1}{2}n}-\rho_{\frac{1}{2}(n-1)}<\frac{4}{\pi}(G_n-G_{n-1}) \qquad (n=1,2,3,\ldots). \tag{12}$$

By summing inequalities of the type (12), we get

$$\rho_{\frac{1}{2}n}-\rho_0<\frac{4}{\pi}(G_n-G_0) \qquad (n=1,2,3,\ldots),$$

and *a fortiori*

$$\rho_{\frac{1}{2}n}<\frac{4}{\pi}G_n \qquad (n=1,2,3,\ldots).$$

This completes the proof of (2).

6. To complete the proof of (3) it is now sufficient to prove that

$$\frac{\rho_{\frac{1}{2}n}}{G_n}<\frac{\rho_{\frac{1}{2}(n+1)}}{G_{n+1}} \qquad (n=1,2,3,\ldots).$$

To effect this object, write (9) in the form

$$\frac{G_{n+1}-G_n}{\rho_{\frac{1}{2}(n+1)}-\rho_{\frac{1}{2}n}}<\frac{G_n-G_{n-1}}{\rho_{\frac{1}{2}n}-\rho_{\frac{1}{2}(n-1)}},$$

so that

$$\frac{G_{n+1}-G_n}{\dfrac{\rho_{\frac{1}{2}(n+1)}}{\rho_{\frac{1}{2}n}}-1}-\frac{G_n-G_{n-1}}{\dfrac{\rho_{\frac{1}{2}n}}{\rho_{\frac{1}{2}(n-1)}}-1}<\frac{G_n-G_{n-1}}{1-\dfrac{\rho_{\frac{1}{2}(n-1)}}{\rho_{\frac{1}{2}n}}}-\frac{G_n-G_{n-1}}{\dfrac{\rho_{\frac{1}{2}n}}{\rho_{\frac{1}{2}(n-1)}}-1}=G_n-G_{n-1}.$$

By summing inequalities of this type, we get

$$\frac{G_{n+1}-G_n}{\dfrac{\rho_{\frac{1}{2}(n+1)}}{\rho_{\frac{1}{2}n}}-1}-\frac{G_1-G_0}{\dfrac{\rho_{\frac{1}{2}}}{\rho_0}-1}<G_n-G_0 \qquad (n=1,2,3,\ldots),$$

that is to say,

$$\frac{G_{n+1}-G_n}{\dfrac{\rho_{\frac{1}{2}(n+1)}}{\rho_{\frac{1}{2}n}}-1}-G_n<\frac{G_1-G_0}{\dfrac{\rho_{\frac{1}{2}}}{\rho_0}-1}-G_0=\frac{5\pi-16}{16-4\pi}<0,$$

and therefore

$$G_{n+1}-G_n<G_n\left(\frac{\rho_{\frac{1}{2}(n+1)}}{\rho_{\frac{1}{2}n}}-1\right) \qquad (n=1,2,3,\ldots),$$

which is equivalent to the desired result.

It may be remarked that the last inequality shows that

$$\frac{\rho_{\frac{1}{2}n}}{G_n}<\frac{\rho_{\frac{1}{2}(n+1)}}{G_{n+1}}<\ldots<\lim_{N\to\infty}\frac{\rho_{\frac{1}{2}N}}{G_N}=\frac{4}{\pi},$$

and this procedure gives an alternative method of proving the result of § 5.

JK

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